Spherical Harmonics

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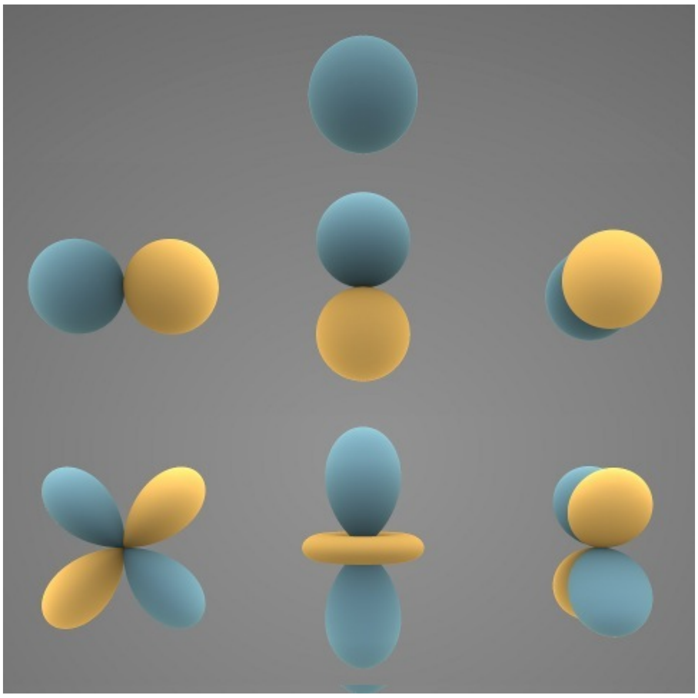
Spherical harmonics are complex mathematical functions that can be added to model the shape changes of an oscillating sphere. Wikimedia Commons has diagrams of the simplest spherical harmonics.

When motion in blue regions is outward in yellow regions it's inward.



Click **Link-1** below this page for the full image.

1 The sphere in line one represents expansion and contraction, the simplest type of oscillation, which is given the symbol "s".

2 The diagrams in line two represent oscillations in a straight line through the centre, that are given the symbol "p". There are three of these, one along each of the x, y and z axes.

3 The diagrams in line three show more complicated higher energy harmonics that we can neglect for the moment.

Visualising spherical harmonics

The distorted 4 kg water balloon below is a retouched frame from a video of a water balloon bouncing on tiles made with my iPhone at 240 fps.

Click **Link-2** below this page and watch the video.

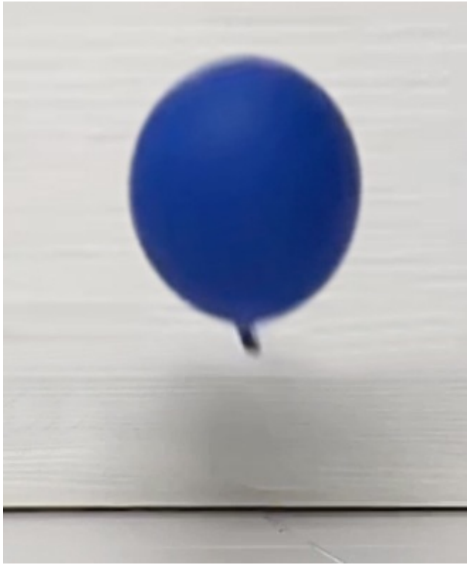
The emerging bumps on the balloon are the sums of spherical harmonics with well known regular shapes and particular frequencies, in the same way that the wriggling of a stretched wire when plucked near one end is the sum of sine-wave vibrations.

Water will not expand and contract like air, there are no simple expansions and contractions of the balloon so there are no spherical "s" modes from the diagram above. *Look closely as the video plays*. Water appears to move backwards and forwards through the centre. The threshing about is not like the random jumping of rats in a bag. It looks to us as to be dominated by "p" modes.

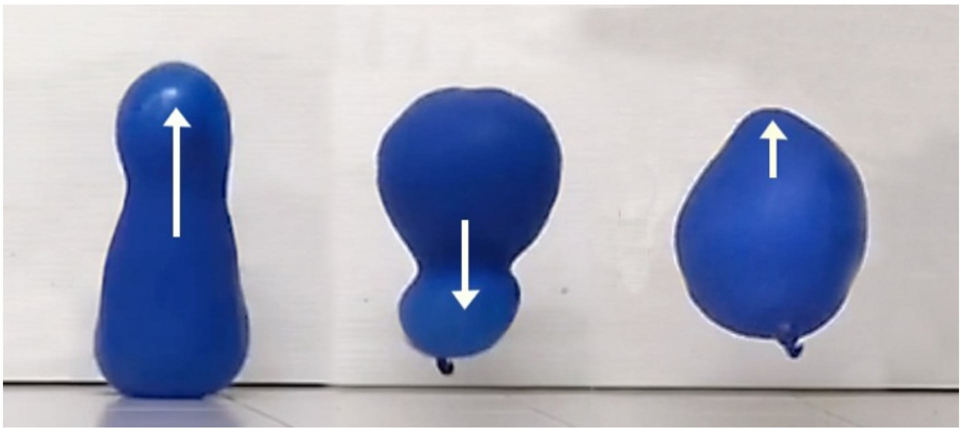
The dominant p modes on a water balloon

Physics types simplify things so they can see and describe what's happening and fit what they see into patterns they find elsewhere.

To simplify the motion of the bouncing balloon we dropped it again with the neck down so that it was more symmetrical around the vertical axis.



Click **Link-3** below this page and watch the video.



Retouched frames 64, 84, and 104 have been repositioned to put the centre of mass of the water at the same height in each frame. Vertical water movement w.r.t. the centre of mass is shown by the white arrows.

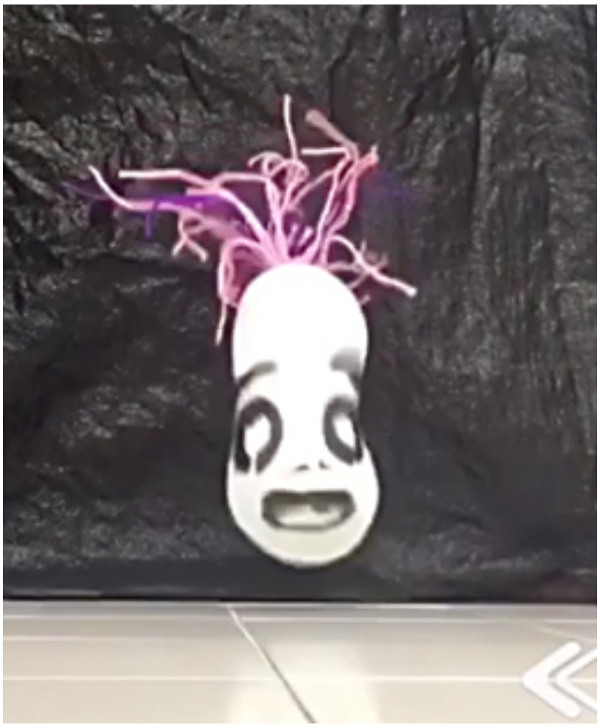
Mode frequency

The damped vertical “p” oscillation marked on the image above is regular, with a period of around $^{40}/_{240}$ = $^{1}/_{6}{}^{th}$ of a second and the frequency is about 6 Hz (cycles per second). No attempt has been made to determine the frequency more accurately. It will depend on the mass of water and the wall tension in the rubber. Thinner rubber will give lower frequencies.

The important thing is that there *is* a frequency and it can be measured. The motions of the balloon are not the random jumping of rats in a bag.

> Note: *most of the kinetic energy of the falling 4 kg balloon is converted to heat as it hits the floor. If you turn up the volume while watching the video, you will hear the impact, but when discussing energy conversion we don’t usually say “heat and sound”. We notice the sound because our ears are very sensitive to vibrations in the air. The kinetic energy converted to sound is a tiny fraction of the energy converted to heat and can be neglected. A larger fraction of the kinetic energy of motion becomes oscillation energy in the balloon. You should mention that if asked.*

A weaker-walled balloon of the same mass is shown bouncing on the right. The p oscillation period is close to 0.25 seconds (f = 4 Hz). The vertical p mode persists and the rubber face rotates around the water due partly to the weight of the hair. The apparent gain of energy with each bounce when the frames are reversed is unfamiliar and looks odd.



Click **Link-4** below this page and watch the video.

The next task …

With the equipment we have it would be easier to find the frequencies of spherical harmonics in the air oscillating inside a regulation basketball bounced on a hard floor. These oscillations can be heard in the room as a high pitched ping just after the bounce. Repeated pings could be recorded with a microphone and the frequencies of the spherical harmonics in the ball found by plotting what is called a Fast Fourier Transform (FFT) on a computer.

Larger balls ring with lower frequencies.

If the air in a ball is replaced with a gas of a different density the frequencies and the character of the oscillations change. A carbon dioxide ball is dead because an atmosphere of pure CO_2 will not transmit sound without rapid energy conversion to heat in the gas, and a ball filled with a lighter gas than air rings with higher frequencies.

The hydrogen atom

Erwin Schrodinger discovered in the 1920's that the possible energy of the single hydrogen atom electron can be *matched* to spherical harmonics. Because electrons were at that time imagined to be orbiting particles the energy levels were called *orbitals*.

> The lowest orbital, filled with two electrons, is the spherical 1s orbital with a point node at the centre of the sphere.
>
> The next possible level of higher energy is the 2s orbital with a point node at the centre and a spherical node between the centre and the surface of the sphere.
>
> The nearest levels of higher energy are the three lowest "p" orbitals, which can each hold two electrons. Their energy is above 2s so they are given the symbol 2p. The 2s and three 2p orbitals (each with two electrons) make up the outer shell of eight in Lewis dot diagrams.

Search for "*diagrams of the orbitals of the hydrogen atom*" on the web.

It was surprising at the time that spherical harmonics could be used in this way. Matching electron energy levels to spherical harmonics in the nano world is nothing like modelling oscillations in the macro world. The hydrogen atom does not behave in any way like a bouncing water balloon.

> *Atomic electron energy does not redistribute to other modes in small amounts so it is not lost to heat after each bounce on a wall. The outer electron shell keeps the energy it has. Bouncing is perfectly elastic and exchanges no heat with the wall, unless the collision is so violent that the electron is moved to a higher empty level. The electron in a higher energy orbital, normally unoccupied, is unstable. Some time later that electron returns to the lower level, emitting light of a wavelength related to the energy difference between the levels. Unexpected, truly amazing in elegance and simplicity, and making no sense at all in the clumsy macro world we we live in. We use Lewis dot diagrams, balls and sticks, paper tetrahedra, and now spherical harmonics to model a reality that we do not directly understand.*

The carbon atom

The carbon atom has one pair of 1s electrons, one pair of 2s electrons, and two unpaired electrons in 2p orbitals. A slight addition of energy promotes one of the 2s electrons to the vacant 2p level and the carbon then has four unpaired bonding electrons. The new model then matches the Lewis picture with four bonding electrons that we've got used to.

A bonded carbon atom is in what we call an excited state, which would normally be unstable and decay with the emission of infrared light, but the bonding energy lowers the overall energy of the electron orbitals much more than the small increase needed to maintain the excited state. The electrons of a bonded carbon atom are stable with a single 2s electron and three single 2p electrons in four occupied orbitals.

Learning about molecular orbital theory, chemical bonding and absorption and emission of light as electrons transition suddenly from one orbital energy to another is a matter of getting used to strange ideas one at a time. Doing physics in a Harry Potter way.

Hagrid has a flying motorbike. Mr. Weasley bewitches a Ford Anglia to fly. We accept the flying car without question and are told that the inside of the car is bigger than the outside. We get used to that idea. Hermione turns up in the last book with a tiny purse that holds the contents of a shipping container: a preposterous notion that we accept because we've had time to get used to it.

Click **Link-5** below this page for fun: to watch the slinky video.