Surds: evaluation

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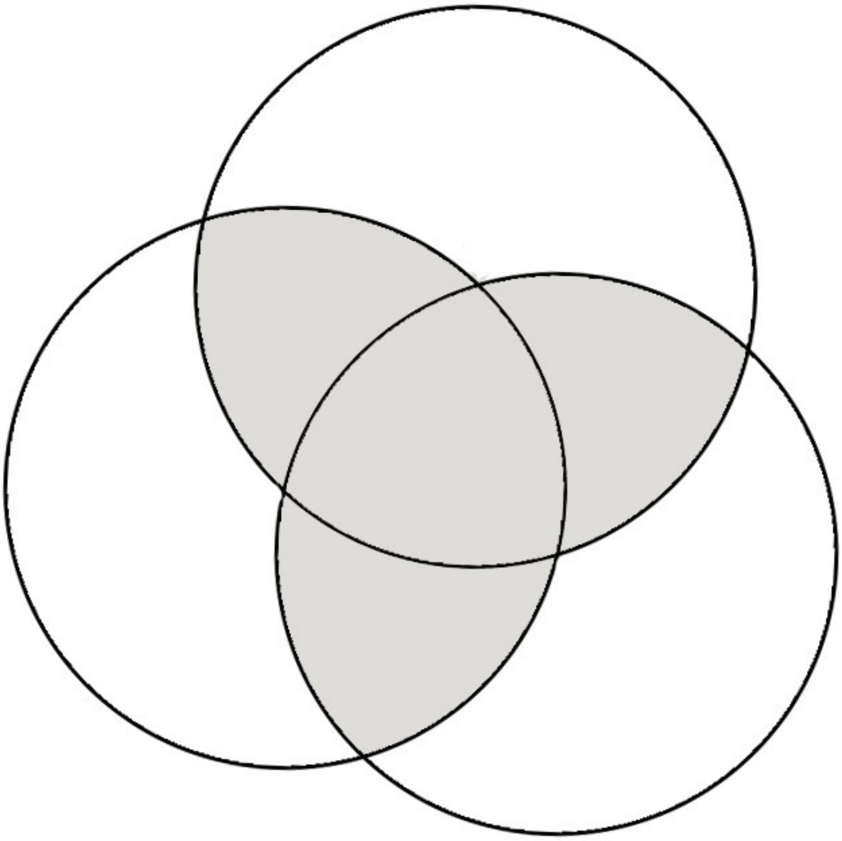
Areas of overlapping circles and triangles can often be expressed in terms of π and either $\sqrt{2}$ or $\sqrt{3}$.



For instance the area A of the unshaded area is given by …

$$A = {}^{r^2}\!/_{2}\,[\pi + 3\sqrt{3}]$$

To write down that area in square cm if the radius of each circle is 4 cm requires a value for π and $\sqrt{3}$. The value of π to three figures is well known and remembered as 3.14, but a value for $\sqrt{3}$ might not be remembered.

Calculators are on phones and the web, but phones can't be sneaked into exams. *Could we find $\sqrt{3}$ to three figures in three minutes*? Of course we could: not with a formula but with a trick that works.

Finding a value for √3 to there figures

1 First guess the value √3 to two figures.

2x2 = 4 … so … √3 will be less than 2. *Let's try 1*.6.

2 The value we want to three figures will be a bit more than 1.6.

Let the difference be x.

$$(1.6 + x)(1.6 + x) \sim 1.6^2 + 1.6x + 1.6x$$

We neglect the x^2 term because x is small so x^2 will be *very small.*

$$\begin{array}{r} 1.6 \\ \underline{\times\ 1.6} \\ \underline{0.96} \\ \underline{1.60} \\ \underline{2.56} \end{array}$$

If you don't know that 16^2 is 256 you have to do a long multiplication.

"Into every life a little rain must fall."

3 Because … $(2.56 + 3.2x) \sim 3$ … [is close to 3] … we can find x.

so … $x \sim (3 - 2.56)/3.2$

$= 44/320$

$= 0.14$ … *doing long division in my head.*

4 So: a more accurate value for √3 is our first estimate (1.6) plus the difference we calculated to be about 0.14.

$$\sqrt{3} = 1.60 + 0.14 = 1.74$$

Written out like that the method looks complicated but it actually isn't.

Summary

1 To find a square root to three figures first guess it to two figures.

2 Let the difference between your guess and a better answer be x.

3 Find x by neglecting the term in x^2 because x^2 is very small.

(*To do that you have to find the square of your guess and do one easy long division that I can do in my head.*)

4 The answer is your guess plus x.

Let's do this once more. Find √2 to four figures.

1 … √2 will be about 1.4 …

2 … and be closer to $(1.4 + x)$ …

3 Find x ... (neglecting x^2 because that is very small) …

$$(1.4^2 + 2\text{x}1.4x) \sim 2$$

$$(1.96 + 2.8x) \sim 2 \qquad \text{… since } (14\text{x}14 = 196)$$

$$x \sim (2 - 1.96)/2.8$$

$$= 4/280$$

$$= 0.014$$

4 … so √2 is close to 1.414

Having done this once by ourselves without a calculator, it's a good idea to remember that value of √2 *so we don't have to do it again.*

Iteration

Iterations are repeated calculations using previous results to get closer and closer to a true value. Iteration is done again and again. We just found the square root of 2 to four figures by hand. If we use that value (1.4141) we can calculate again and get an answer correct to six or perhaps seven figures and so on. That would require one long multiplication and one long division each time. We *could* do it by hand: but to try that in an exam takes up too much time. There is another way. Use a spread sheet on a computer to do the multiplications and divisions.

The program we use for physics experiments and measurements is Logger pro from vernier.com. A user defines *manual columns* for data inputs and *calculated columns* to find what they want with equations and the data.

We set up two manual columns: one to enter the numbers (n) and one labelled IT_1 (Iteration 1) to enter initial guessed square root values. We create a series of calculated columns to find corrections that we call Δx_1, Δx_2 … etc. and a series of columns IT_2, IT_3, … etc. for the calculated square root approximation after each iteration.

A table with four iterations to five significant figures.

	Square root calculation									
	n	IT_1	Δx_1	IT_2	Δx_2	IT_3	Δx_3	IT_4	Δx_4	IT_5
1	3	1.6	0.1375	1.7375	-0.005	1.7321	-0.005	1.7266	0.0000	1.7266
2										
3										

The expression for Δx_1 looks like this …

Name: Δx_1

("n" – "IT_1"*"IT_1")/(2*"IT_1")

The expression for IT_2 looks like this …

Name: IT_2

"IT_1"+"Δx_1"

After one iteration in the table above, we see that the calculated square root (IT_2) for √3 is 1.74 to three figures, which matches the value found by hand with one calculation beginning with 1.6 as a first estimate.

Entering guesses (approximations) for the square roots of the integers from 1 to 9 gives the following iterations.

1	1.0000	0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
2	1.4000	0.0142	1.4143	-0.000	1.4142	-0.000	1.4141	0.0000	1.4141
3	1.7200	0.0120	1.7321	0.0000	1.7321	0.0000	1.7320	0.0000	1.7320
4	2.0000	0	2.0000	0.0000	2.0000	0.0000	2.0000	0.0000	2.0000
5	2.2000	0.0363	2.2364	-0.000	2.2361	-0.000	2.2358	0.0000	2.2358
6	2.4000	0.0500	2.4500	-0.000	2.4495	-0.000	2.4490	0.0000	2.4490
7	2.6000	0.0461	2.6462	-0.000	2.6458	-0.000	2.6453	0.0000	2.6453
8	2.8000	0.0285	2.8286	-0.000	2.8284	-0.000	2.8283	0.0000	2.8283
9	3.0000	0	3.0000	0.0000	3.0000	0.0000	3.0000	0.0000	3.0000

Numbers from 1 to 9 on the left and their square roots to four decimal places on the right.

The value entered as IT_1 for √3 in the second table is 1.72 which is a more accurate first estimate of the value of √3 than the 1.6 used above. The right-hand column has 1.7320, which is the value returned by an online calculator truncated to five figures.

Simple repeated calculation (enough times) gives the square root of any number to any desired number of decimal places.

Useful numbers

Maths and Physics examiners don't normally expect students to memorise physical constants, square roots (surds) and the like, but there is no rule (prohibition) against doing so. We find it helpful for instance, to know that the area of an equilateral triangle in triangular units is the length of a side squared and that the conversion factor from that to square units is $\sqrt{3}/4$.

My father remembers a Physics examiner at his University who left out a required number (the charge on the electron) because he felt that a third-year Honours student could be expected to know that. Dad did: but some did not and were embarrassed by that.

The following numbers should be familiar: in particular those in bold.

$$\mathbf{13^2 = 169}$$
$$\mathbf{14^2 = 196}$$
$$\mathbf{15^2 = 225}$$
$$\mathbf{16^2 = 256}$$
$$17^2 = 289$$
$$18^2 = 324$$
$$19^2 = 361$$

$\boldsymbol{\pi = 3.14}$

$$\mathbf{\sqrt{2} = 1.41}$$

$$\mathbf{\sqrt{3} = 1.73}$$

$$\mathbf{\sqrt{5} = 2.24}$$

$$\mathbf{\sqrt{6} = 2.45}$$

$$\sqrt{7} = 2.65$$

$$\sqrt{8} = 2.83$$

Knowing one or more of these is often helpful: for instance, to write down the area with which we began this discussion.

Appendix 1

There are other ways to find square roots by iteration. There are almost always other ways to do anything in mathematics. This method is not new. It was found by the Babylonians in antiquity and was described by Heron of Alexandria. It goes like this.

Let's guess the square root of 9 as 4. We know that's too large, but that's what we want. Because 4 is too large 9/4 will be too small. A better approximation would be the average of 4 and 9/4 so let's call that better approximation x_1 and calculate it.

$$x_1 = {}^{(4 + 9/4)}/_{2}$$

$$= {}^{6.25}/_{2}$$

$$= 3.125$$

Do that again using x_1 as the new approximation.

$$x_2 = {}^{(3.125 + 9/3.125)}/_{2}$$

$$= 3.0025$$

Do that again using x_2 as the new approximation.

$$x_3 = {}^{(3.0025 + 9/3.0025)}/_{2}$$

$$= 3.00000104$$

This method converges very quickly and if the initial estimate is high subsequent estimates will always be high. In our example just three iterations have given the square root of 9 correct to six figures even when the first guess was very approximate. The calculations are not difficult but do require long divisions. As an exercise you could set up a spread sheet in Logger pro (or Excel) to do the calculations, and then decide which method you might use to find √**7** by hand.

Appendix 2

There is an obscure reason to teach simple iterations in High School. Computers, phones and digital watches do so many different things that we forget they're just adding machines. Siri is done with logic circuits that add two binary numbers (expressed as 1's and zeros) and return the result to a register. That's it: the addition of numbers, two at a time.

Huawei watch: an accelerometer counts steps and monitors the hours of sleep. Heart beats are counted optically. Run distance and speed are checked with gps locations: all by addition.

Subtraction is done in a computer by adding the 2's compliment [more of that in a future post]. Square roots require what we call multiplication: that's just repeated addition, and division: that's just repeated subtraction. Clever people at Intel have put together logic circuits on a single chip, purpose built and interconnected to output square roots: by addition, not magic.

Computers are *fast* adding machines. Compared to your brain they work at unbelievable speed. If in some distant time we make a computer that we believe to be conscious (when we've worked out what we mean by that) she would probably class us all as being dead, based on the speed of our processing equipment (our brains).