Triangular cm

Shannon and Ian Jacobs

I was asked at school to: "*Make a model illustrating mathematics*."

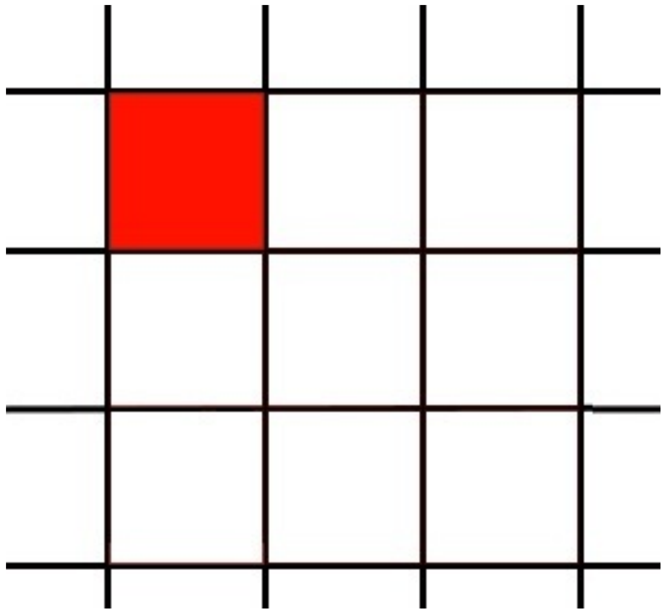
I made a bedroom with models of furniture and wallpapers printed from patterns I drew with a tessellation program.

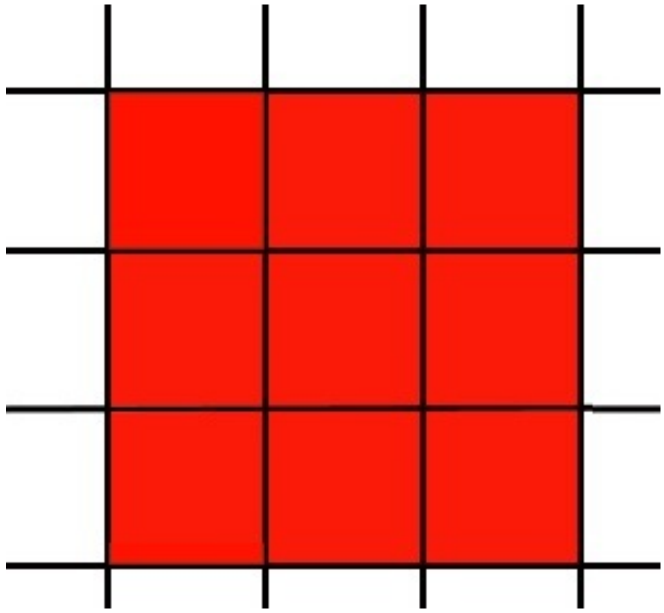
https://www.mathsisfun.com/geometry/tessellation-artist.html

A tessellation is a repeating space-filling pattern of shapes. The simplest are tilings using just one shape: squares, rectangles, triangles or hexagons.

Dad was helping me make the large patterns to print from the small sections I had drawn. Doing that reminded him of something mathematical he wanted to show me.

Covering a flat object with one cm *squares* and counting them is one way to define *area*. Because squares are always used most people think that area can only be measured in square units.

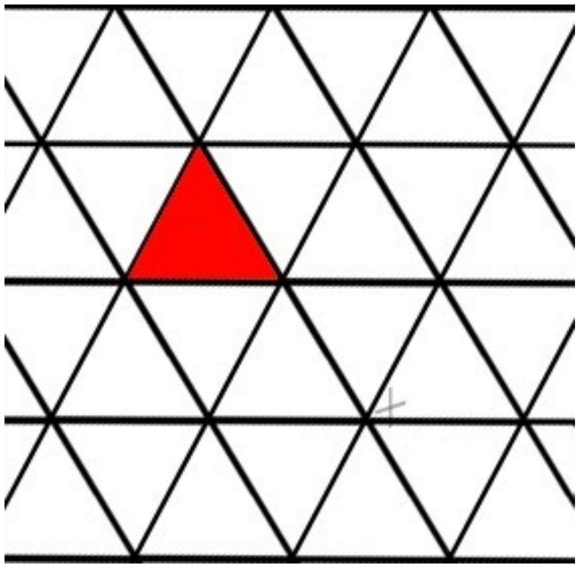


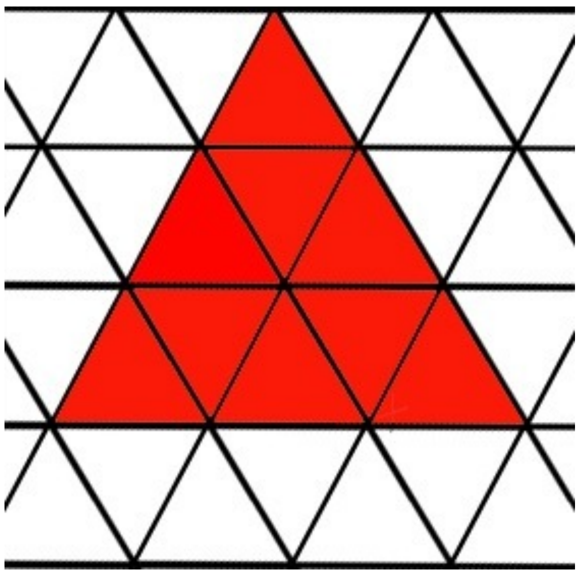


The large red square has an area of 3x3 = 9 cm^2

We say three *squared* is nine and forget that we are really talking about the area of a square that's three cm on a side.

Squares *tessellate* the plane. Counting squares defines an area. Because many of our shapes are bordered by straight lines and have right angle corners, square units are convenient, but equilateral triangles also tessellate the plane.



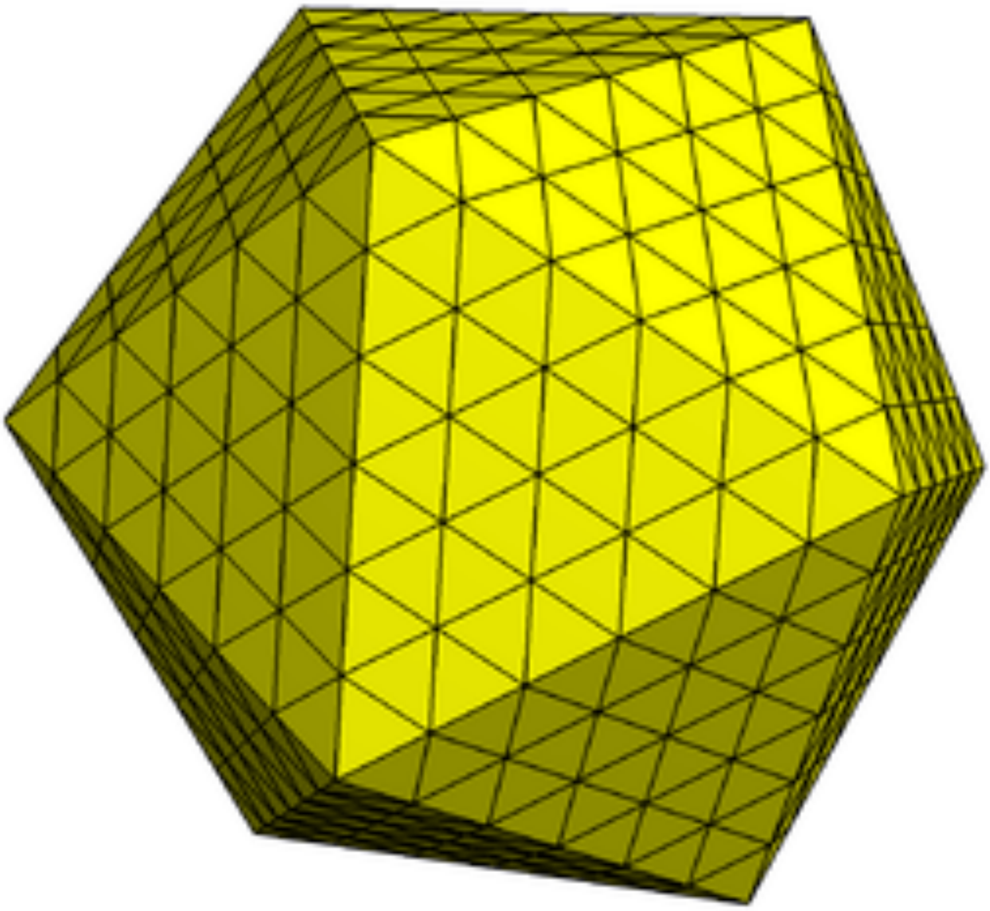


The large red triangle has an area of 3x3 = 9 triangular cm

Squaring the length of a side of the large equilateral triangle counts the triangular units of area. The relationship is the same as that for a a tessellation of squares and holds for any area that’s tessellated with self-similar shapes.

Triangular centimetres can sometimes be a convenient measure of area.

For instance: suppose we are asked to find the surface area of a twenty sided icosahedron with edges of 6 cm.



Tomruen, CC BY-SA 4.0 <https://creativecommons.org/licenses/by-sa/4.0>, via Wikimedia Commons

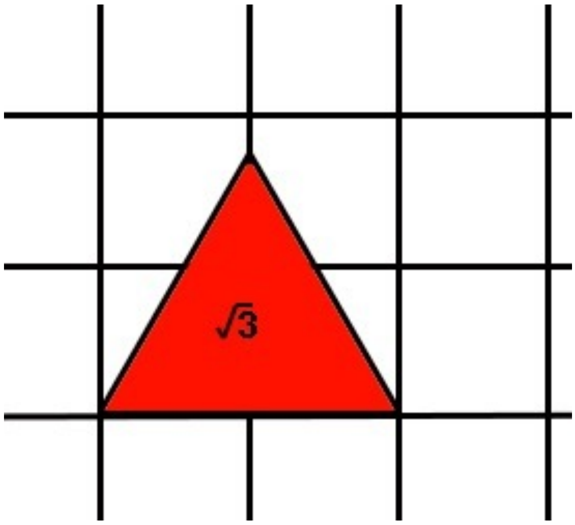
The surface area *A* is given at once as …

$$A = 20 \times 6^2$$

$$= 720 \text{ triangular cm}$$

An examiner might not be very happy with that answer. The mark schedule will have the area in *square* cm so I could only use the triangular cm calculation if I knew how to convert triangular cm to square cm.

Converting triangular centimetres to square centimetres

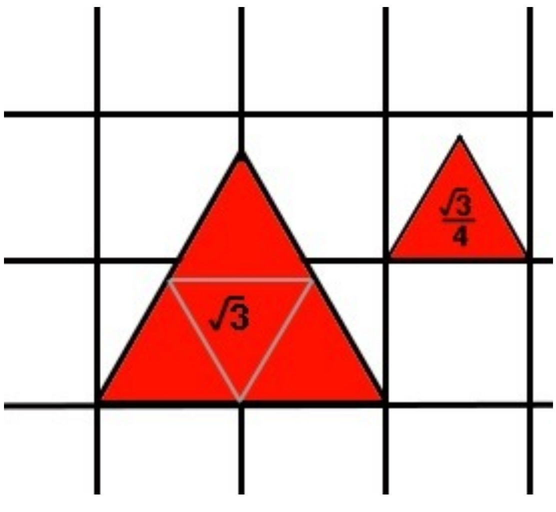


If I knew the height of the equilateral triangle in cm I could use half base times height to find the area, but all I know is that the sides are all 2 cm.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

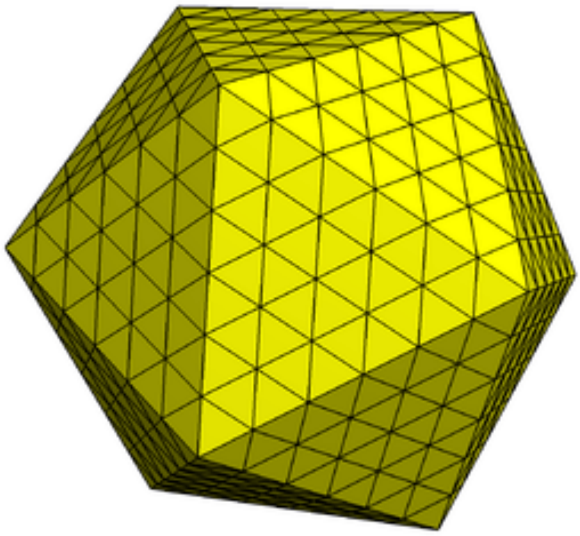
… where s is the semi-perimeter
and a, b and c are the lengths of the sides.

Because, for this triangle, $s = 3$ and $a = b = c = 2$, the area is $\sqrt{3}$.



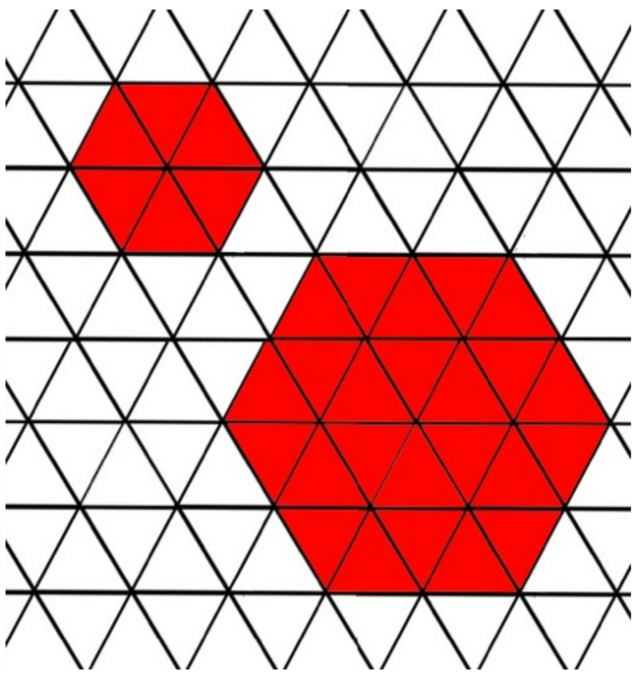
The area of one triangular cm is $\sqrt{3}/4$ square cm.

Surface area in square cm.



The surface area of the icosahedron is $720^{\sqrt{3}}/_4$ square cm.

Hexagons



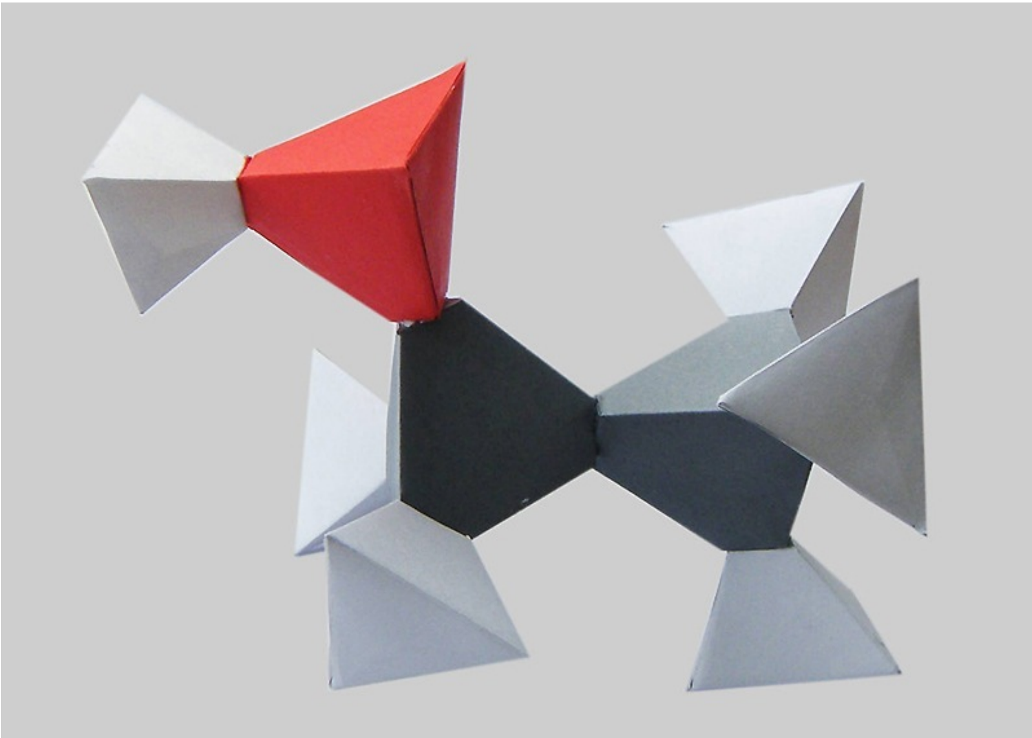
The area of the smaller hexagon is 6 triangular cm … or $3^{\sqrt{3}}/_2$ square cm.

The area of the larger hexagon is $6x2^2$ triangular cm … or $6\sqrt{3}$ square cm.

Appendix (for experts)

Unit equilateral triangles (what we call triangular cm) is a measure of area not mentioned in traditional mathematics books and seldom if ever used in practice. They are a mathematical curiosity, useful to create understanding of what an area measure is, but of little practical value. Mathematics is littered with things like this, clever, interesting for their own sake, but having no useful applications at the time of development.

It happens that we do have a use for them. We make molecular models with modified tetrahedra. The models have faces that are equilateral triangles with one or more snub points (a smaller equilateral triangle removed from a vertex). The surface areas of these unusual models can be written down in triangular units and converted to the usual square units with the conversion factor $\sqrt{3}/4$.

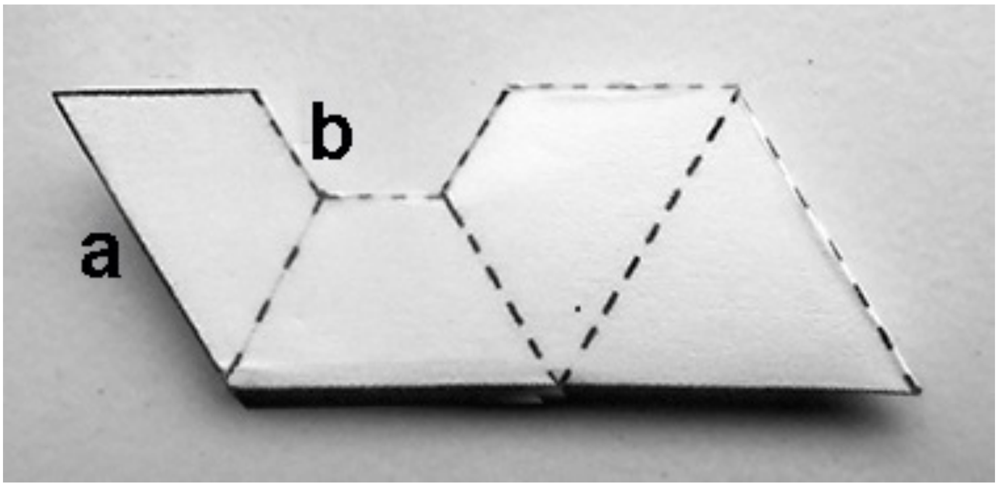


A model of an ethanol molecule C_2H_5OH.

The surface area of this three dimensional model would be difficult to calculate directly in square units, but it can be written down in triangular units in terms of three lengths: a, b, c. *You could try that for yourself after reading the examples below.*

Examples

1 The figure shows the net of a regular tetrahedron with one snub point. Lengths are marked **a** and **b**.



i Find the total area of the exposed paper in terms of a and b.

ii Write down the surface area of the snub tetrahedron that can be made with this net.

Solutions

i In triangular units the area is given by …

$$A = 4a^2 - 3b^2$$

The area in square units is …

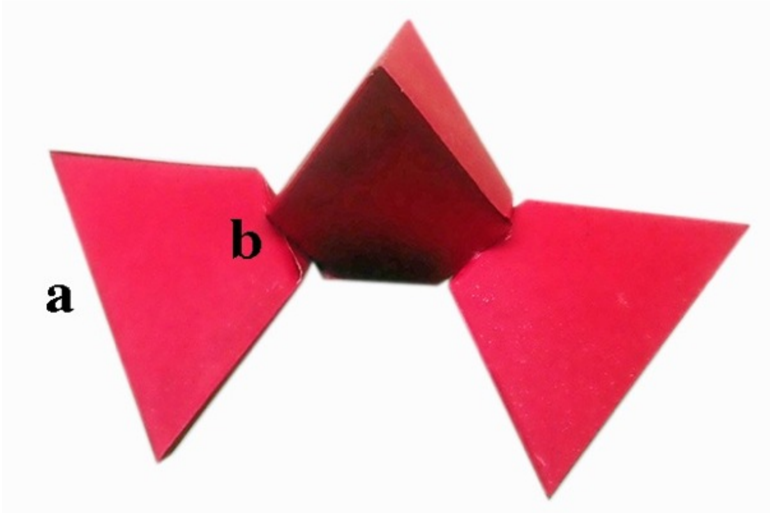
$$A = \sqrt{3}/_4[4a^2 - 3b^2]$$

ii In triangular units the base of the tetrahedron has area a^2, the three sides have area $(a^2 - b^2)$ and the snub point has area b^2.

The surface area in square units is …

$$A_s = \sqrt{3}/_4[a^2 + 3(a^2 - b^2) + b^2] = \sqrt{3}/_2[2a^2 - b^2]$$

2 The symmetrical model is made with three regular tetrahedra, one with two snub points and two with just one each.



Two lengths are marked a and b.

i Find the surface area in terms of a and b.

ii Find the volume in terms of a and b.

Solutions

i In triangular units the surface area A_s is the sum of two equilateral triangles of area a^2, 10 snub equilateral triangles of area $(a^2 - b^2)$ and 6 small equilateral triangles exposed at the two glued joints (red in the figure at right) that add to $2(^2/_3 b^2)$.

In triangular units … $A_s = 2a^2 + 10(a^2 - b^2) + 2(^2/_3 b^2)$

In square units … $A_s = {}^{\sqrt{3}}/_4[12a^2 - 8^2/_3 b^2]$

ii The volume in cubic units of a regular tetrahedron with edge x is $^{x^3}/_{6\sqrt{2}}$

The volume $= 3a^3/_{6\sqrt{2}} - 4b^3/_{6\sqrt{2}} = (3a^3 - 4b^3)/_{6\sqrt{2}}$