Area of a quadrilateral

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To find a relationship that gives the area of several special quadrilaterals it's helpful to begin with the simplest of all quadrilaterals: a square.

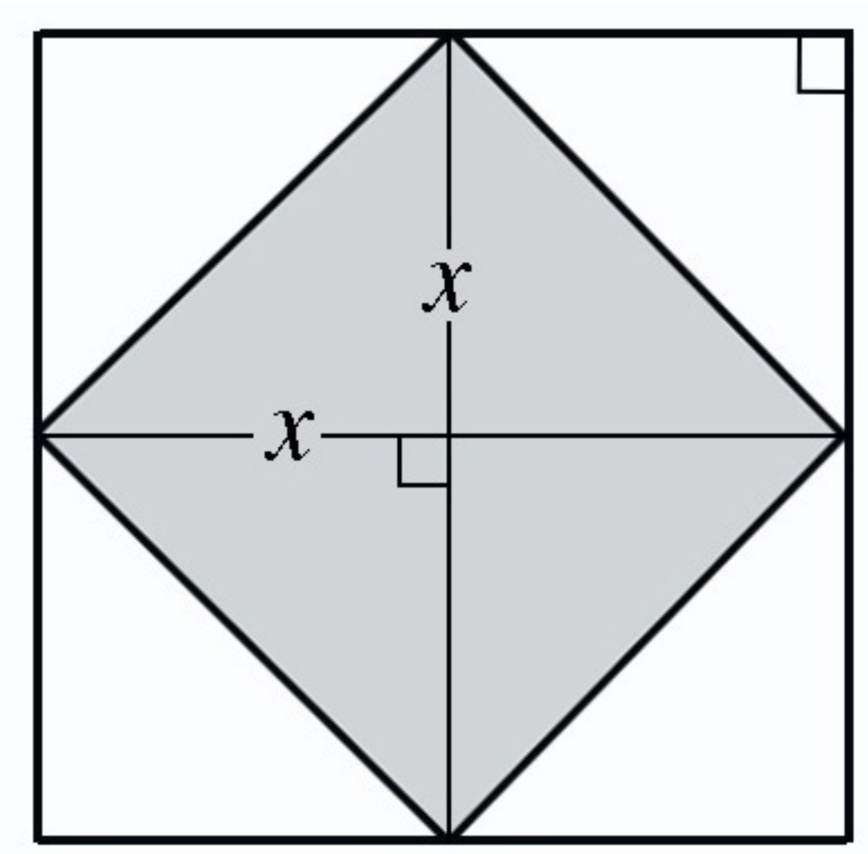


Figure 1. A square within a square

The outer square has sides of length x. Right angles and the diagonals of the inner square are marked on the diagram.

Each of the four outer white triangles can be rotated or flipped to exactly cover the area of the inner grey square. The area of the grey square is half the area of the outer square (x^2).

$$\text{Area} = {}^1/_2\, x^2$$

The area of a square is given by half the product of the diagonals.

Suppose we stretch the diagram by lengthening the x axis.

The outer square becomes a rectangle of dimensions x and y.

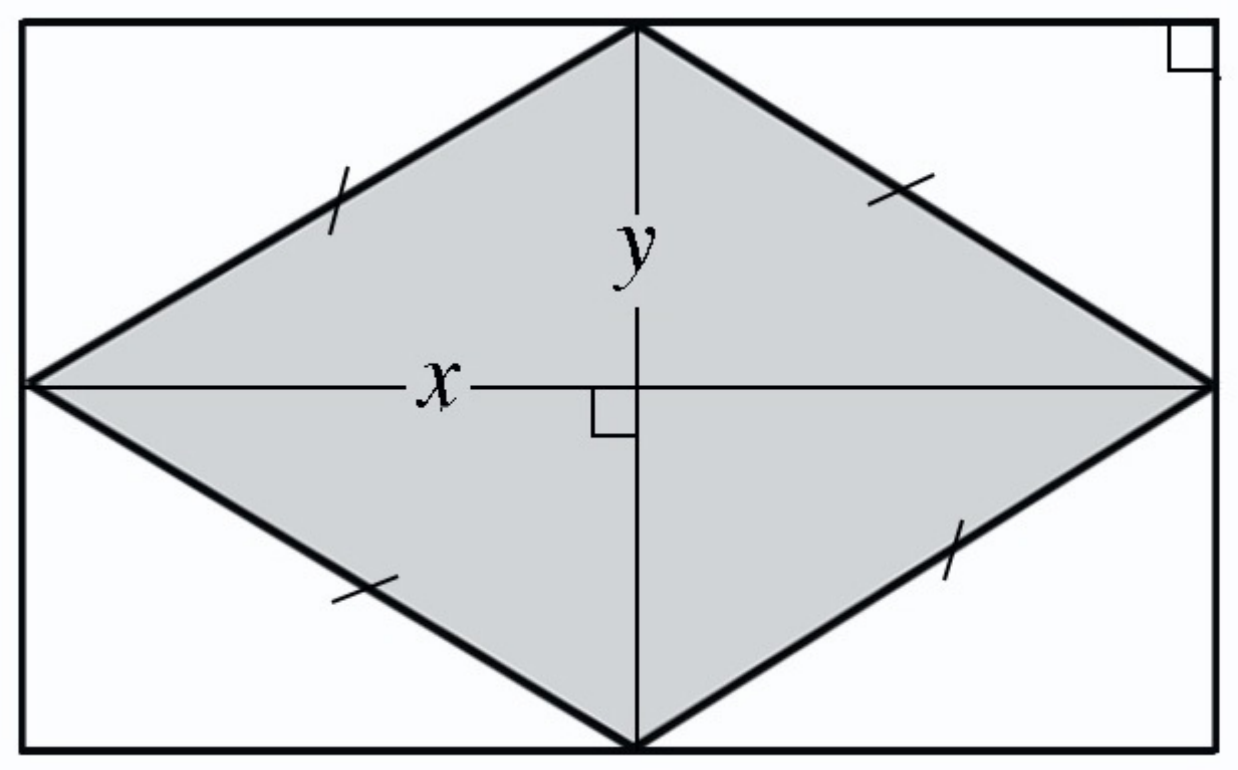


Figure 2. A rhombus within a rectangle.

The lengths of the sides of the inner figure (a rhombus) remain the same, the right angles remain and the square has become a rectangle.

> *A rhombus of any size and shape can be drawn on perpendicular diagonals by adding pairs of parallel lines of equal length. Diagonals of that rhombus intersect at right angles by construction.*

Because the four outer white triangles can each be rotated to exactly cover the inner grey area, the area of the rhombus is half the area of the outer rectangle (xy).

$$\text{Area} = {}^{1}/_{2}\, xy$$

The area of a rhombus is given by half the product of the diagonals.

Suppose we modify the figure of the rhombus by scaling the x axis for points to the right of the vertical diagonal.

The outer rectangle has dimensions x (which is reduced in length) and y as above.

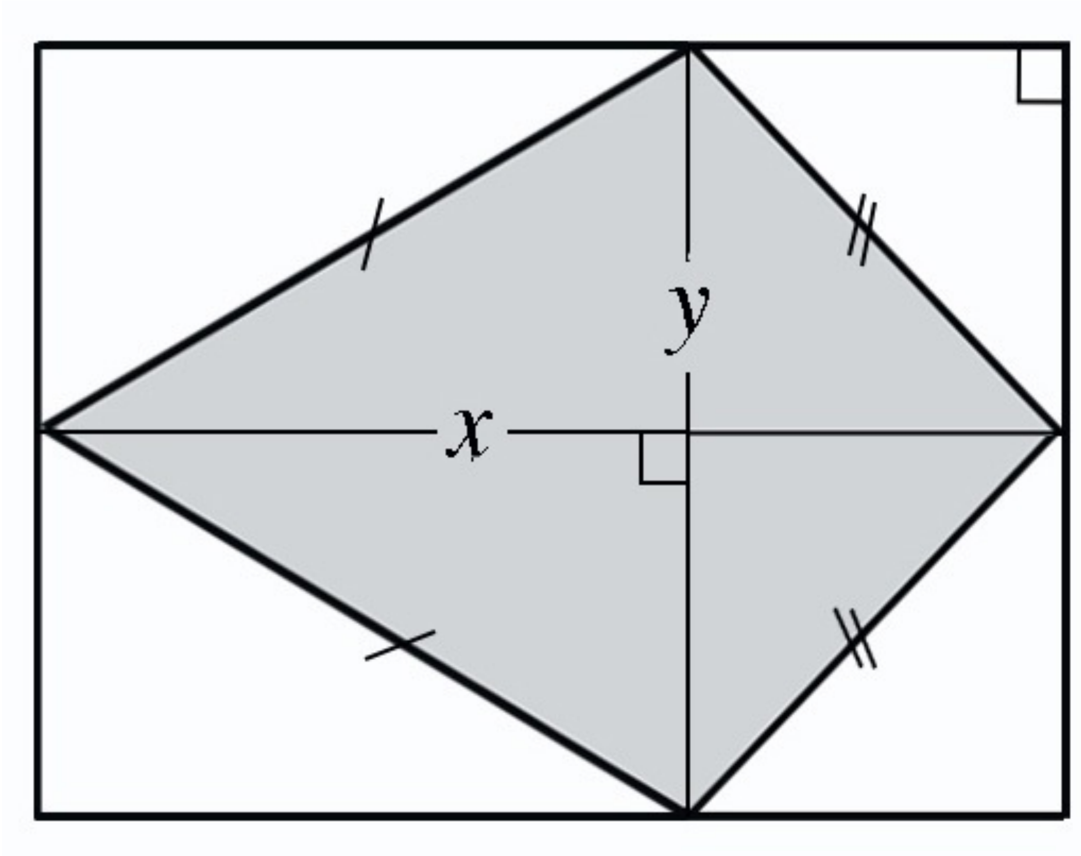


Figure 3. A symmetrical kite within a rectangle.

The kite has two pairs of equal sides, on the left and on the right of the vertical diagonal. The diagonals intersect at right angles.

Because the four outer white triangles can each be rotated to exactly cover the inner grey area, the area of the symmetrical kite is half the area of the outer rectangle (xy).

$$\text{Area} = {}^1/_2\, xy$$

The area of a kite is given by half the product of the diagonals.

Suppose we now modify the kite by scaling the *y* axis below the x axis.

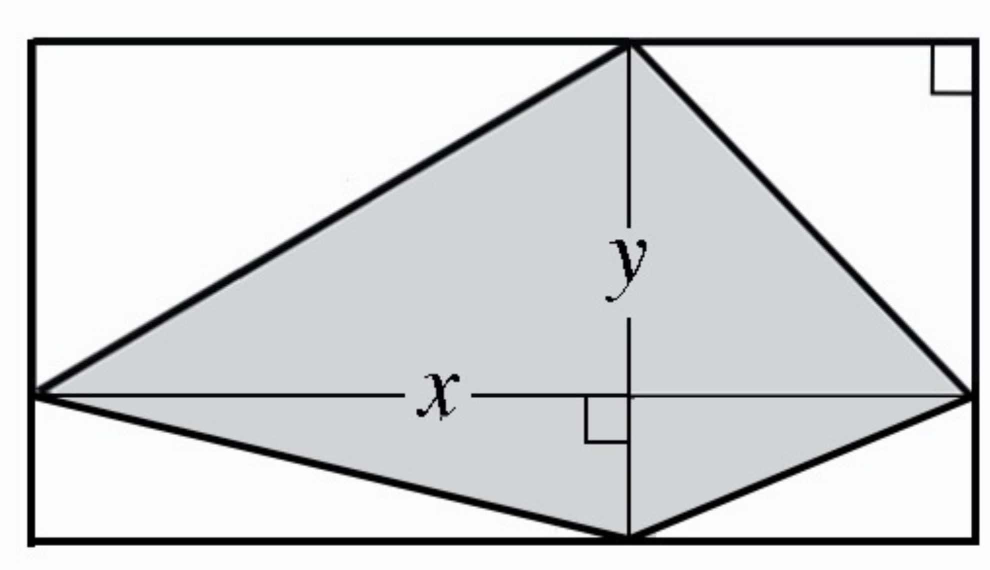


Figure 4. A quadrilateral with diagonals that cross at right angles.

The lengths of the sides of the quadrilateral are unequal and the diagonals intersect (by construction) at right angles. The same arguments apply.

The area of the quadrilateral is $^1/_2\,xy$ (half the area of the rectangle).

Finally: if we scale the *y* axis to zero below the *x* axis, we are left with a triangle.

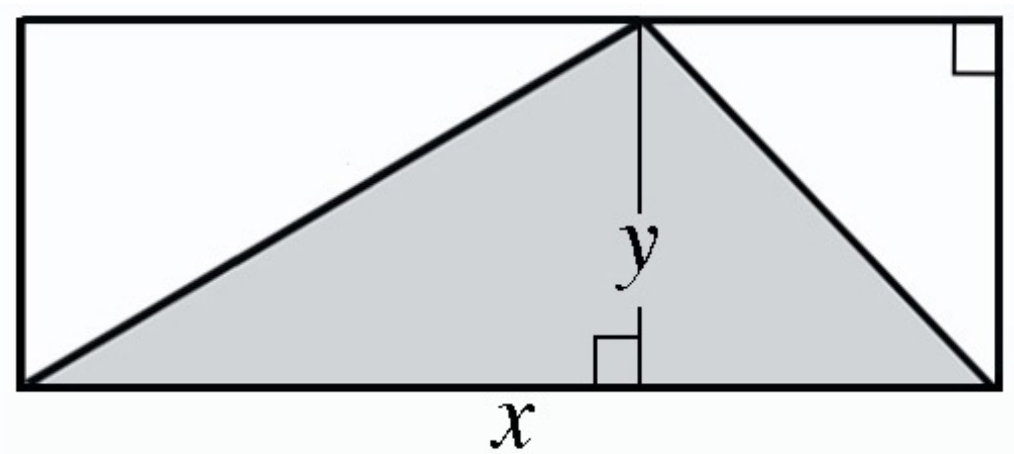
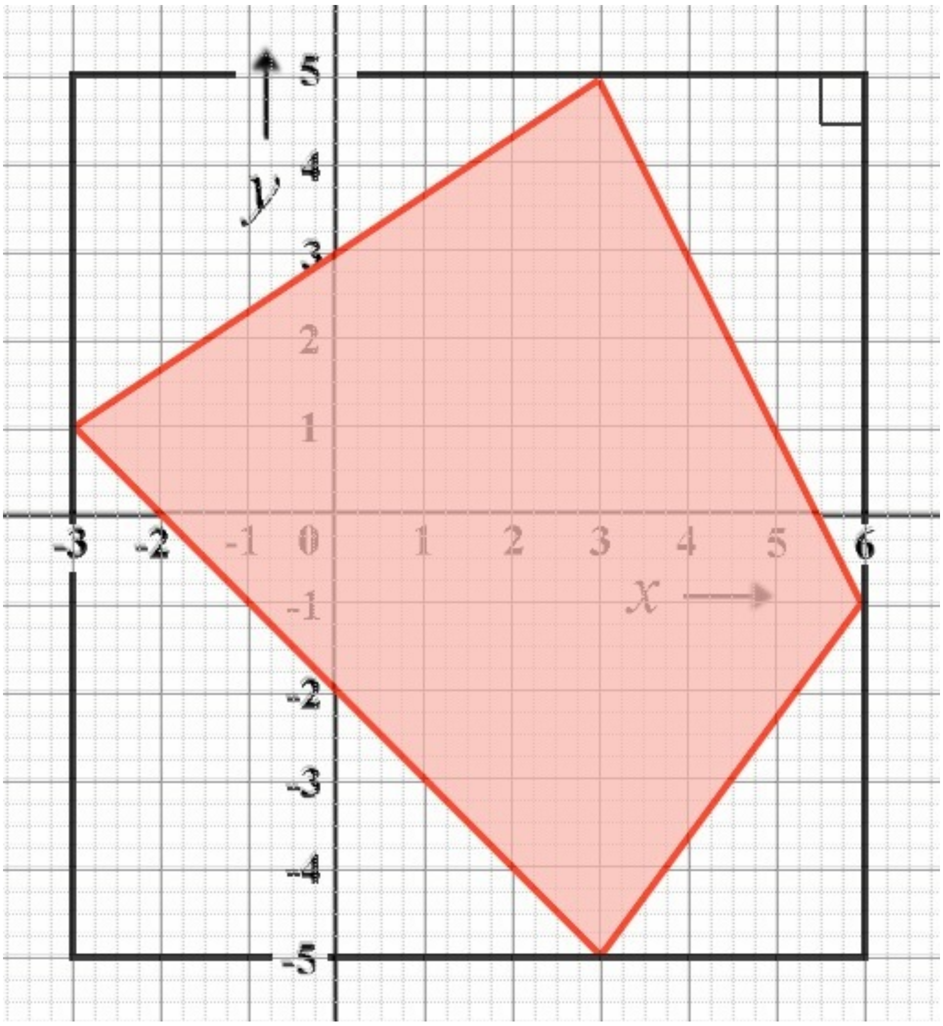


Figure 5. A triangle with base *x* and altitude *y*.

Area = $^1/_2\,xy$ … half the area of the rectangle *xy*.

If the diagonals of any given quadrilateral intersect at right angles the area is half that of a surrounding rectangle. If the diagonals don't intersect at right angles we could find the area of two triangles but …

… if we know the coordinates of each vertex there is an easier way.



Construct a surrounding rectangle. The areas of the four outer white triangles are easily found and the quadrilateral follows.

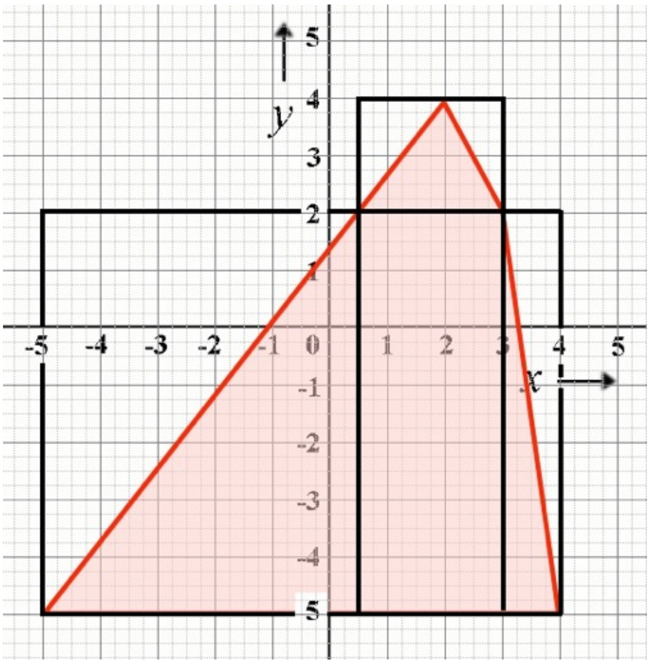
The area of the quadrilateral is …

$$90 - (18 + 6 + 9 + 12) = 45 \text{ sq. units.}$$

Questions like this have been arranged to have simple outcomes.
If you ask your own question by putting four random points on the plane to make the corners of a quadrilateral you will find that some require more rectangles.

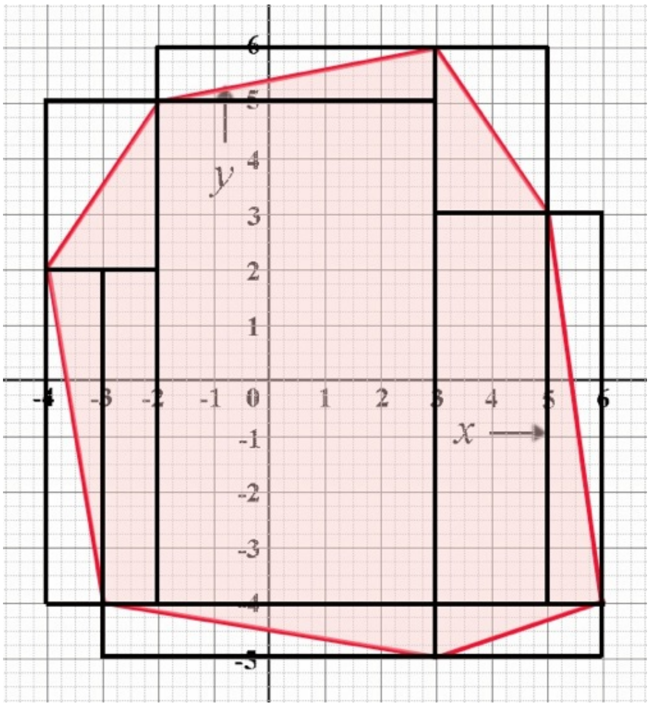
Appendix

The quadrilateral below can be solved with two rectangles.



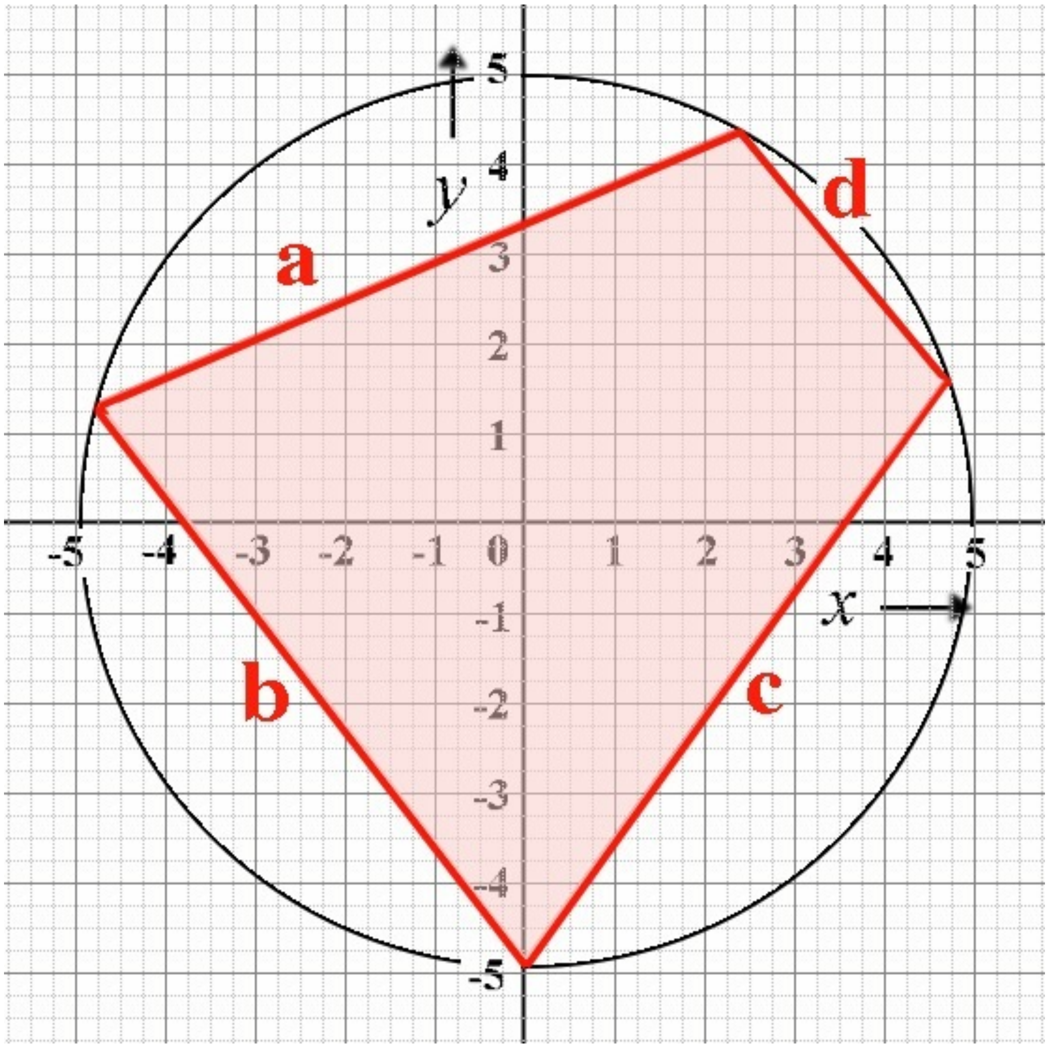
Area = (21+1.5+1+3.5) + 10.5 = 37.5 sq. units

The area of *any* convex polygon can be found in this way.



The area = (3+3+3+3+3+3.5+1.5) + (6+45+14) = 85 sq. units

A triangle is a special case of a quadrilateral with perpendicular diagonals that has one diagonal reduced in length to form the altitude of a triangle. In a similar way Heron's formula for the area of a triangle could be written down as a special case if we knew the area of a quadrilateral with known sides a, b, c and d.



Remarkably the area of a cyclic quadrilateral with known sides was discovered in India by *Bhramaputra*, born in 648 AD.

$$\text{Area} = \sqrt{[(s - a)(s - b)(s - c)(s - d)]} \ldots$$

… where s is the semi-perimeter $^1/_2(a + b + c + d)$

Reducing d to zero gives Heron's formula for the area of a triangle.

$$\text{Area of a triangle} = \sqrt{[s(s - a)(s - b)(s - c)]}$$