Circles: Area and Circumference

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There is a short passage in the Hebrew Bible from around 500 BC that gives π, the ratio C/D of a circular pool, as $^{30}/_{10}$, which equals 3.

From the Latin: King James Version: 1611: 1 Kings 7:23
And he made a molten sea, ten cubits from the one brim to the other: ... height was five cubits: and a line of thirty cubits did compass it about.

That was a good guess: but measure it carefully both ways.

The ratio is closer to 22/7.

That is 3.14 as a decimal number to three figures but this too is not exact.

C/D for a circle is known to be an *irrational number* that cannot be written as the *ratio* of two whole numbers. We have now found it to *trillions* of decimal places. *Don't stay awake in bed thinking about why we did that*.

For a discussion of the Biblical reference see …

https://www.abarim-publications.com/Bible_Commentary/Pi_In_The_Bible.html

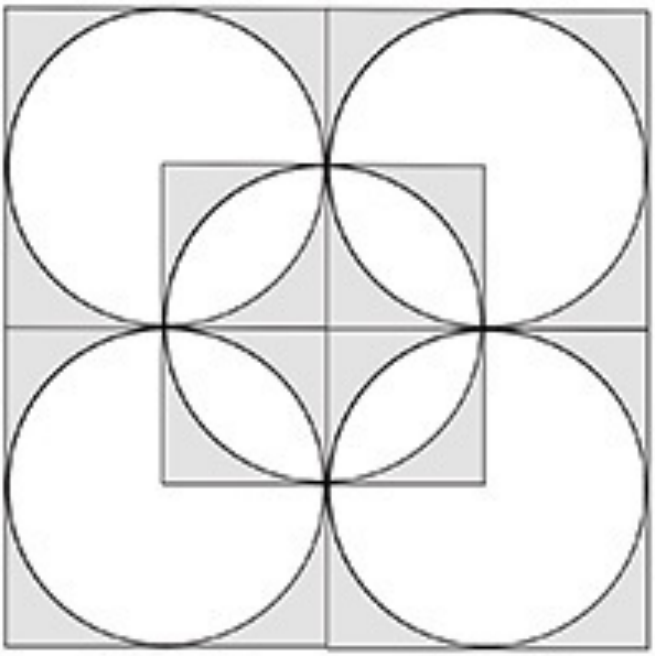
The circumference of a circle is $2\pi r$

The area if a circle is πr^2

Remember these two.

Examples

Exam questions must be attempted without a calculator. They are written carefully, often with dimensions that simplify calculation.



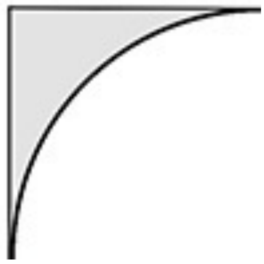
1 A square is 28 cm on a side. It is inscribed with five squares that are 14 cm on a side and five circles of radius 7 cm as shown.

i Find the total area of the shaded portion.

ii Find the total length of the curved lines.

Note: take advantage of the strange dimensions to simplify calculation by taking π as $22/7$.

We could find just one area and one curve length and then count.



The area A of the shaded portion of the small square is the area of the square (7x7) minus the area of one quarter of a circle of radius 7.

$$A = (7\text{x}7) - (^{22}/_7\ \text{x}7\text{x}7)/4$$

$$= 7(7 - 5.5)$$

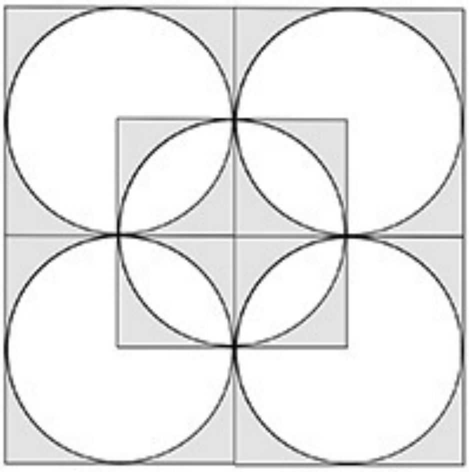
$$= 10.5\ \text{cm}^2$$

The length L of the curve above is one quarter of the circumference.

$$L = (2\text{x}^{22}/_7\text{x}7)/4$$

$$= 11\ \text{cm}$$

Counting gives the shaded area and curve length.



Shaded area $A = 20\text{x}10.5$

$= 210\ \text{cm}^2$

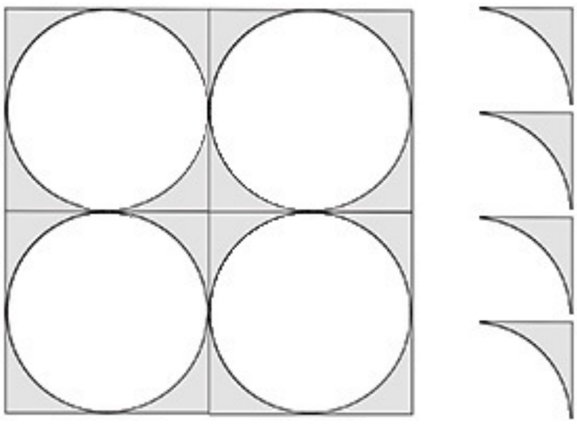
Curve length $L = 20\text{x}11$

$= 220\ \text{cm}$

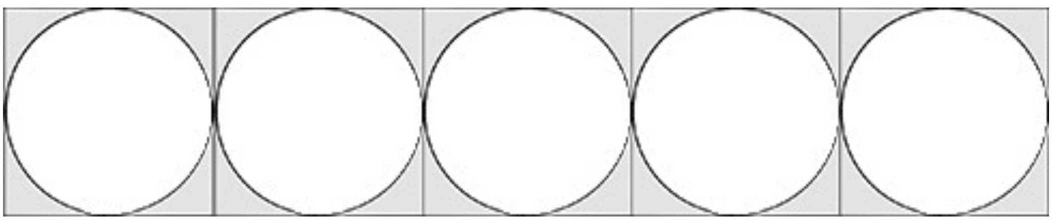
Question 1 continued

Our solution is disappointing. It seems to us to be clumsy. Is there a more elegant satisfying way to do this? In mathematics we find that there is always another way.

Let's rearrange the pieces.



… becomes …



The shaded area A = $5(14\text{x}14 - {}^{22}/_{7}\text{x}7\text{x}7)$

$= 5\text{x}7(28 - 22)$

$= 30\text{x}7$

$= 210\ \text{cm}^2$

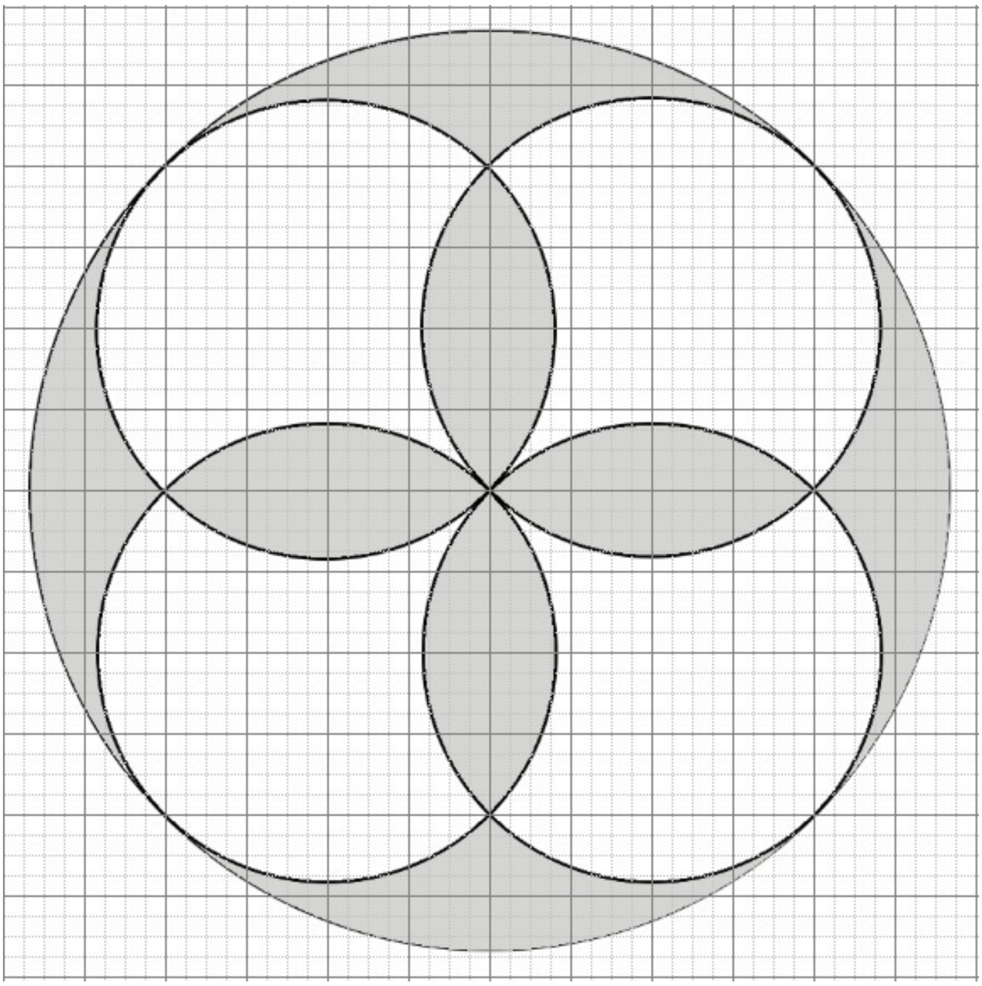
The curve length L = $5\text{x}2\text{x}{}^{22}/_{7}\text{x}7$

$= 220\ \text{cm}$

This working seems to us to be more memorable and satisfying.

Examiners have been making up circle area questions for years and years. Publishers protect their work. Diagrams cannot just be taken from exam papers without permission, but ideas are not subject to copyright. We ask our own questions and ask for answers in terms of π and a radius r.

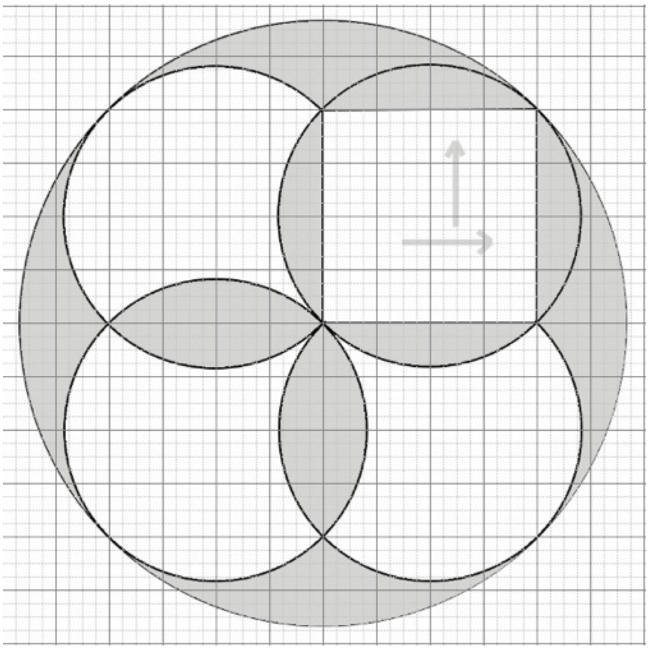
2 Four identical circles within a circle are drawn on a square grid. Each small inner square is 2.5 mm on a side.



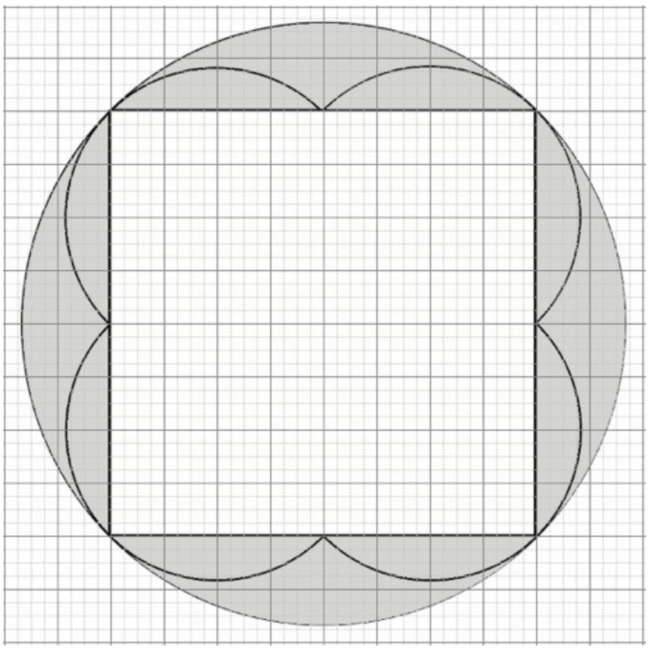
i Simplify the calculation of the shaded area A by modifying the diagram in some way. Express A as simply as you can without numerical calculation.

ii Write down an expression with the least possible number of terms for the length, L, of the four curved lines of a central petal.

Close examination of the diagram shows that halves of the central petals can be moved to compete a square.



Complete the large central square.



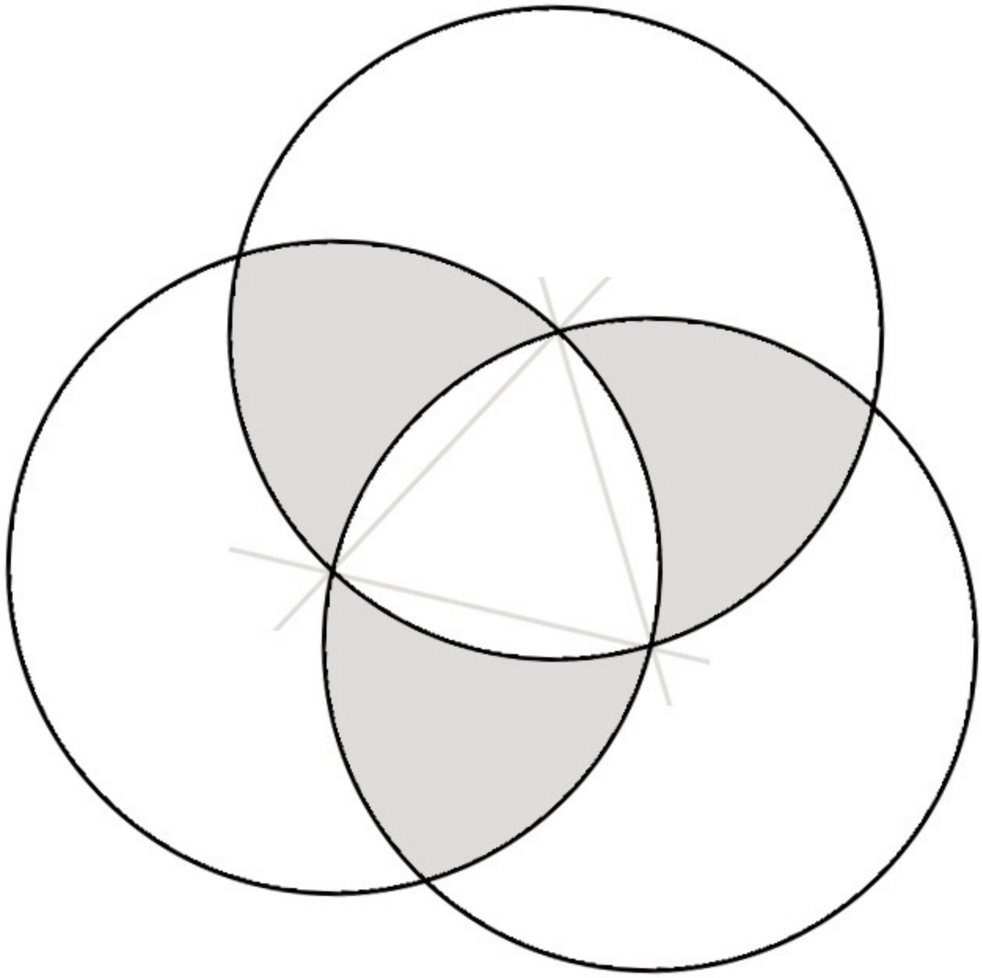
The radius of the outer circle is $4\sqrt{2}$ and of an inner circle is $2\sqrt{2}$.

i *S*haded area $A = \pi (4\sqrt{2})^2 - 8\text{x}8 = 32(\pi - 2)$ cm^2

ii The length $L = 2(2\pi\, 2\sqrt{2}) = 8\sqrt{2}\pi$ cm

3 Questions with three overlapping circles often involve equilateral triangles. This one is clever and instructive.

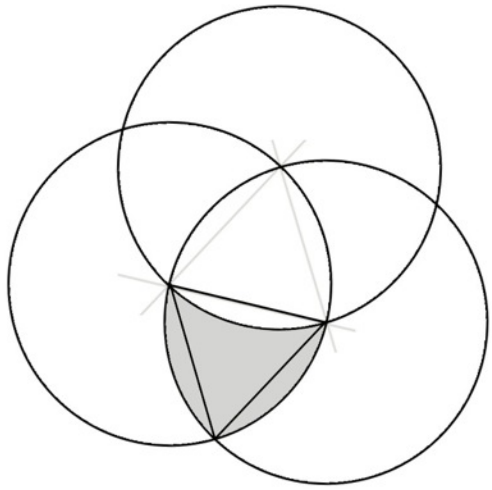
> Three identical circles are drawn with their centres at each vertex of an equilateral triangle. The sides of the triangle are the radii of the circles.



> ***i*** Find the area A of the shaded portions.
>
> ***ii*** Find the perimeter L of the inner bloated triangle.

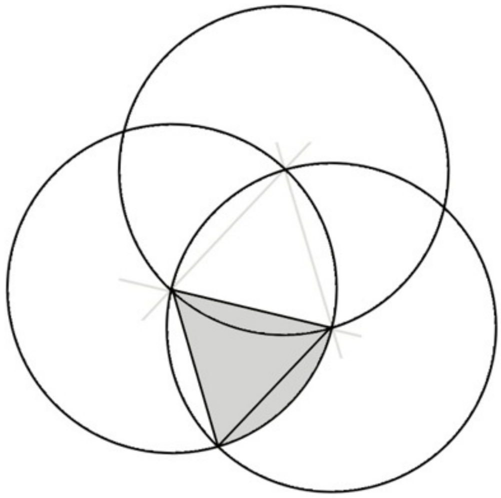
Before we attempt this question, look carefully at the diagram.
It has been drawn with strong conditions on the placement of the circles. We will need to use that information to solve for the areas.

A second equilateral triangle is drawn with sides that are each radii of one of the circles.



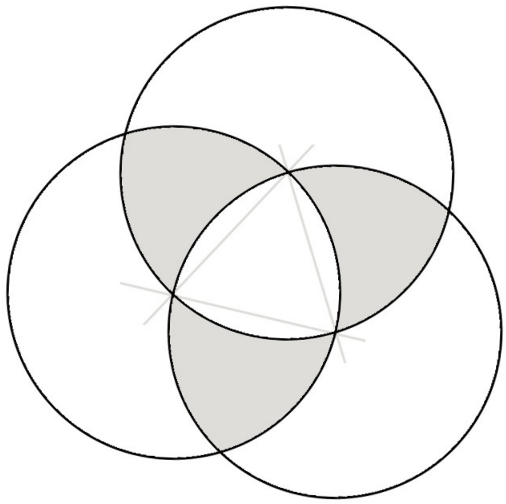
Altering the shading converts the area required into a segment of the lower left-hand circle. The shaded area is one sixth the area of a circle.

$A_1 = {}^1/_6\, \pi r^2$



The three shaded areas together will be half the area of a complete circle.

$A_2 = {}^1/_2\, \pi r^2$



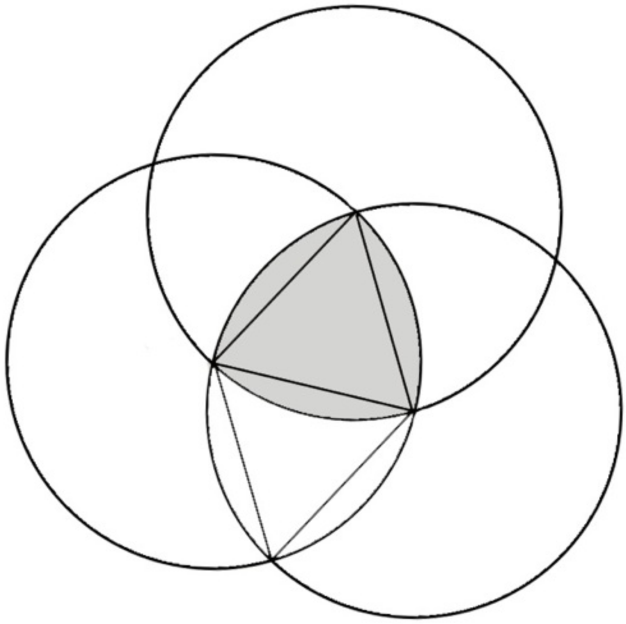
i The shaded area $A = {}^1/_2\, \pi r^2$ cm^2

ii One side of the inner *Reuleaux triangle,* is one sixth of the circumference of a circle.

$L = {}^1/_2\, \pi r$ cm

The Reuleaux triangle

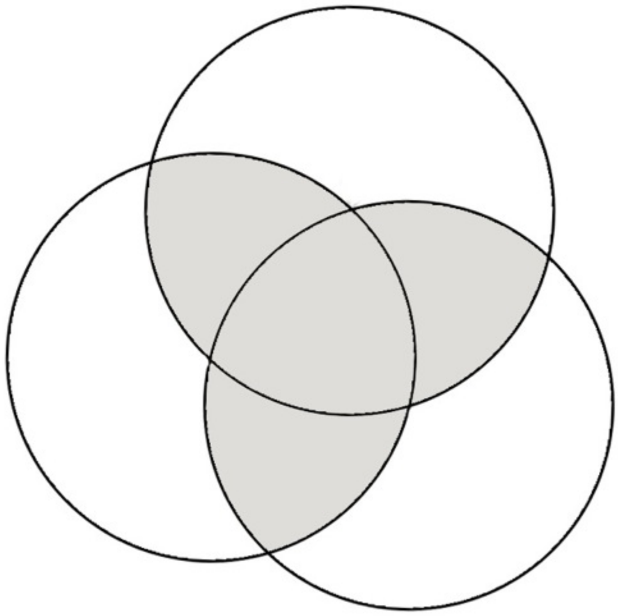
The shaded area can be thought of as three overlapping sectors minus two triangles. The area of the triangle in triangular units is r^2. That is converted to square units by multiplying by $\sqrt{3}/4$.



$$A_3 = A_2 - 2\,\sqrt{3}/4\, r^2$$

$$A_3 = 1/2\, \pi r^2 - 2\sqrt{3}/4\, r^2$$

The total shaded area A_4 can now be written down in terms of π and r.



$$A_4 = A_2 + A_3$$

$$= \pi r^2 - 2\sqrt{3}/4\, r^2$$

We can now find the area of the unshaded portion of the diagram.

The area of all three circles if separated would be $3\pi r^2$. Because the circles overlap, the area will be less than $3\pi r^2$ by $(2A_2 + 3A_3)$.

$$A_5 = 3\pi r^2 - 2A_2 - 3A_3$$

$$A_5 = 3\pi r^2 - \pi r^2 - 3[1/2\, \pi r^2 - 2\sqrt{3}/4\, r^2]$$

$$= 1/2\, \pi r^2 + 3\sqrt{3}/2\, r^2$$