Logical Geometrical Arguments

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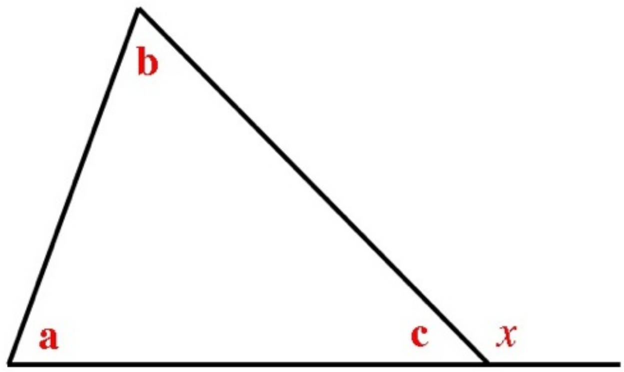
We argue from the known to the unknown. We can seldom do that in one step so we go in smaller steps that we understand. I imagine myself just skipping the steps. Dad says he's never met a student who could do that.

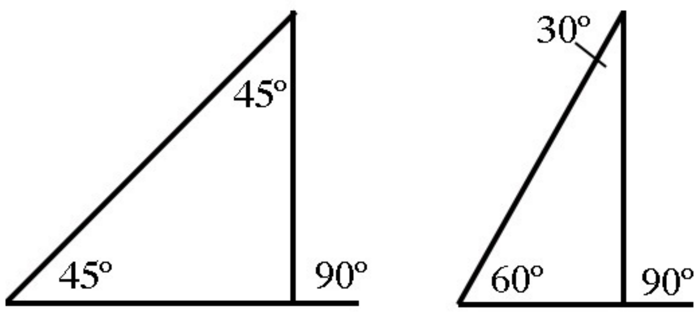
1 Triangles

We begin with what we know.



The interior angles of a triangle add to 180.

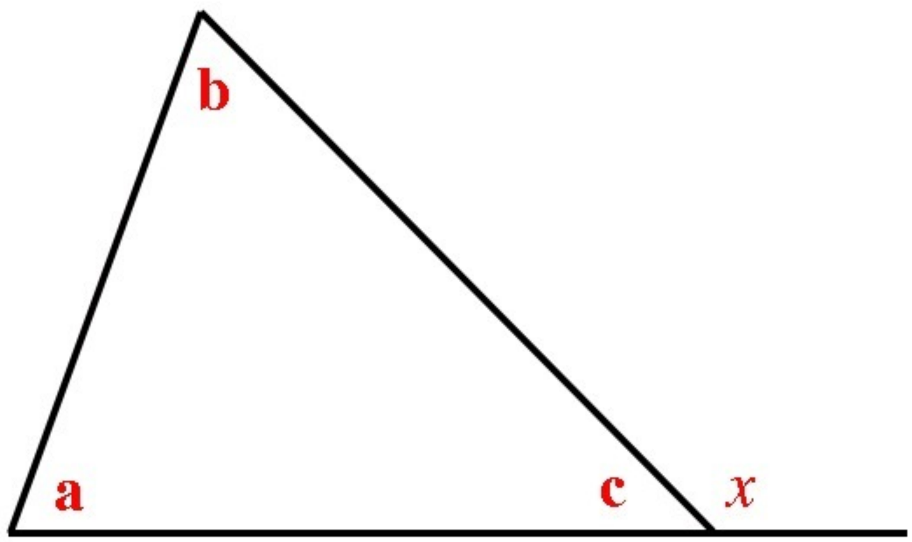
$$a + b + c = 180$$



The exterior angle x is marked on the diagram.
For these triangles the exterior angle x is equal to a + b.

Is that true for any triangle?

Proving that x = a + b for any triangle



a + b + c = 180 … (sum of interior angles) … [1]

c + x = 180 … (angles on a straight line)

… so … x = 180 - c … [2]

Adding 1 and 2 … x = a + b + c - c

… so … x = a + b

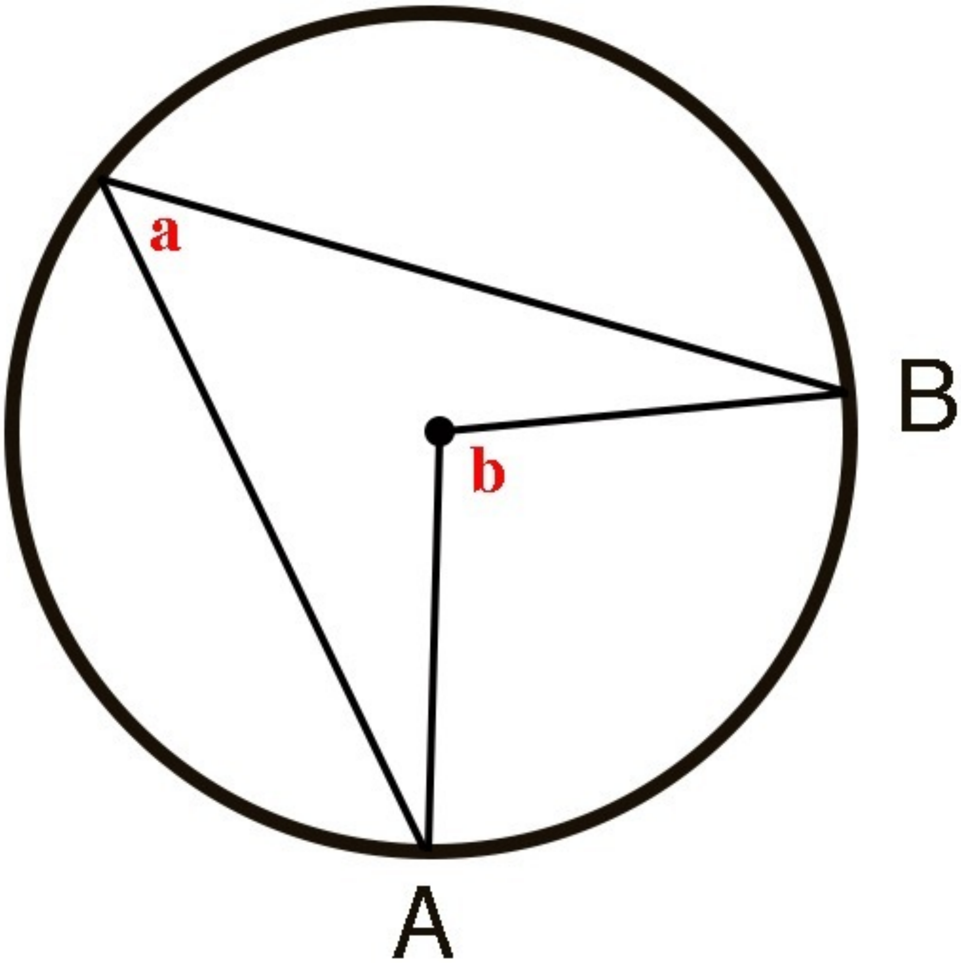
The exterior angle (x) equals the sum of the opposite interior angles (a + b) in any triangle. If we had drawn diagrams carefully and measured angles with a protractor, x would have been equal to (a + b) within a degree or so, but we know now that they *are* equal, exactly and always: there can be no mistake.

Why are we doing this?

This, and selected two-step examples that follow, are taught in schools to develop logical reasoning, which we are not naturally good at.

2 Subtended angles

A and B are points on a circle.



Angle a is said to be *subtended* at the circumference.

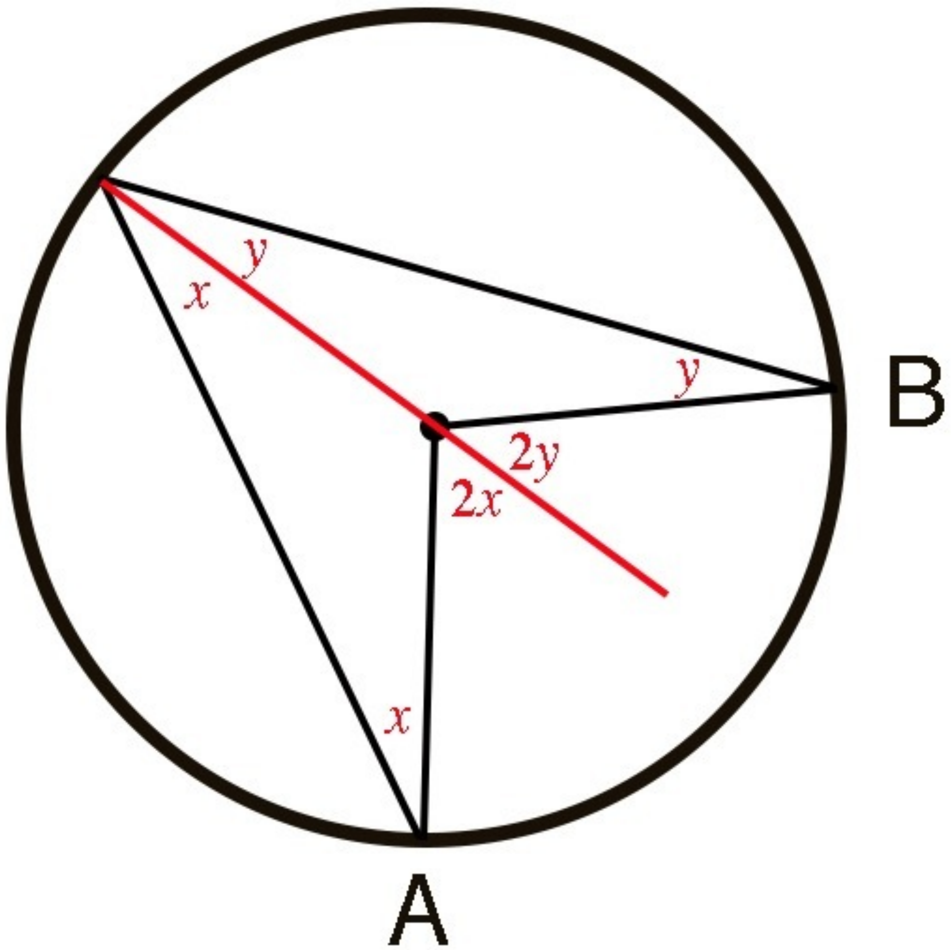
Angle b is said to be *subtended* at the centre.

Angle a is clearly not equal to angle b.

If we measured the angles we might find that very nearly b = 2a.

To prove that for any diagram like this we must use things we know and argue from there. We know two things about angles and triangles. The trick is to add a line to the diagram so we can use what we know.
We add a line to the diagram to form two triangles with exterior angles.

Equal angles in the isosceles triangles and their exterior angles are marked on the diagram.



Write down what we know …

$a = x + y$

… and … $b = 2x + 2y$

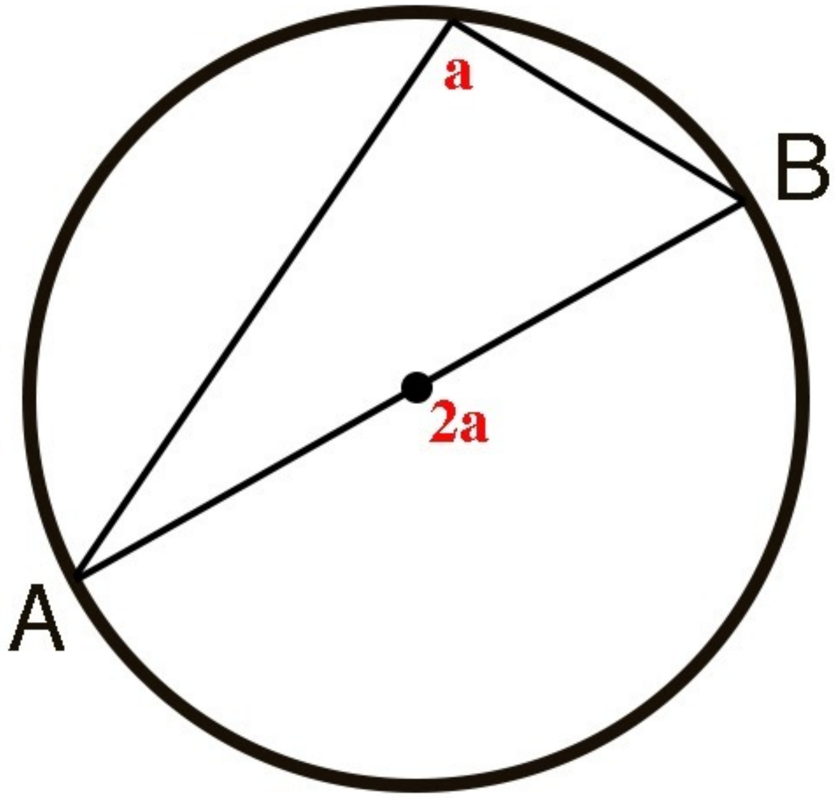
… so … $b = 2(x + y)$

$= 2a$

Because the diagram was not drawn in any special way it follows that the angle subtended at the centre from any two points A and B on a circle is equal to twice the angle subtended at the circumference.

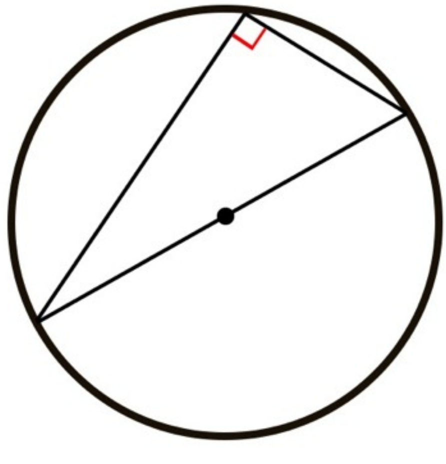
3 Angle subtended on a diameter

The line AB is the diameter of the circle.

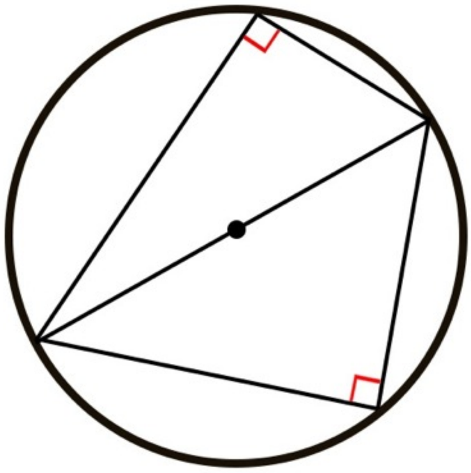


The angle subtended at the circumference is labelled a
… and the angle subtended at the centre is labelled 2a.

The diagram was not drawn in any special way. It follows that any angle subtended at the circumference on a diameter of a circle is a right angle.



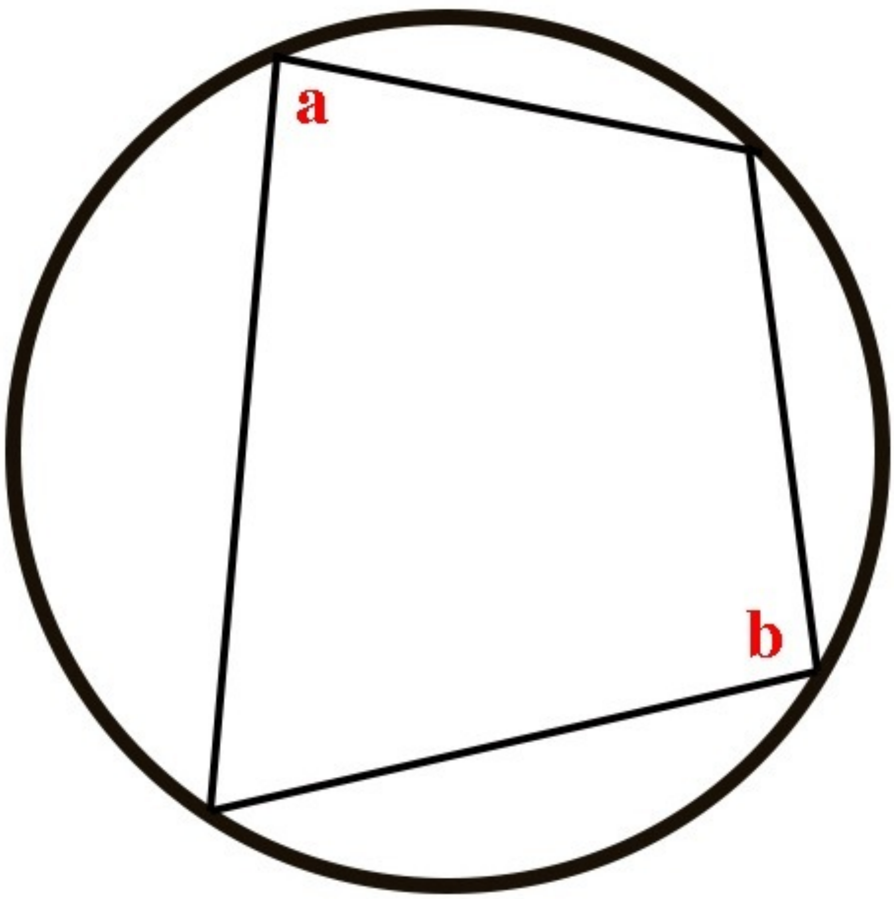
4 Opposite angles of a cyclic quadrilateral



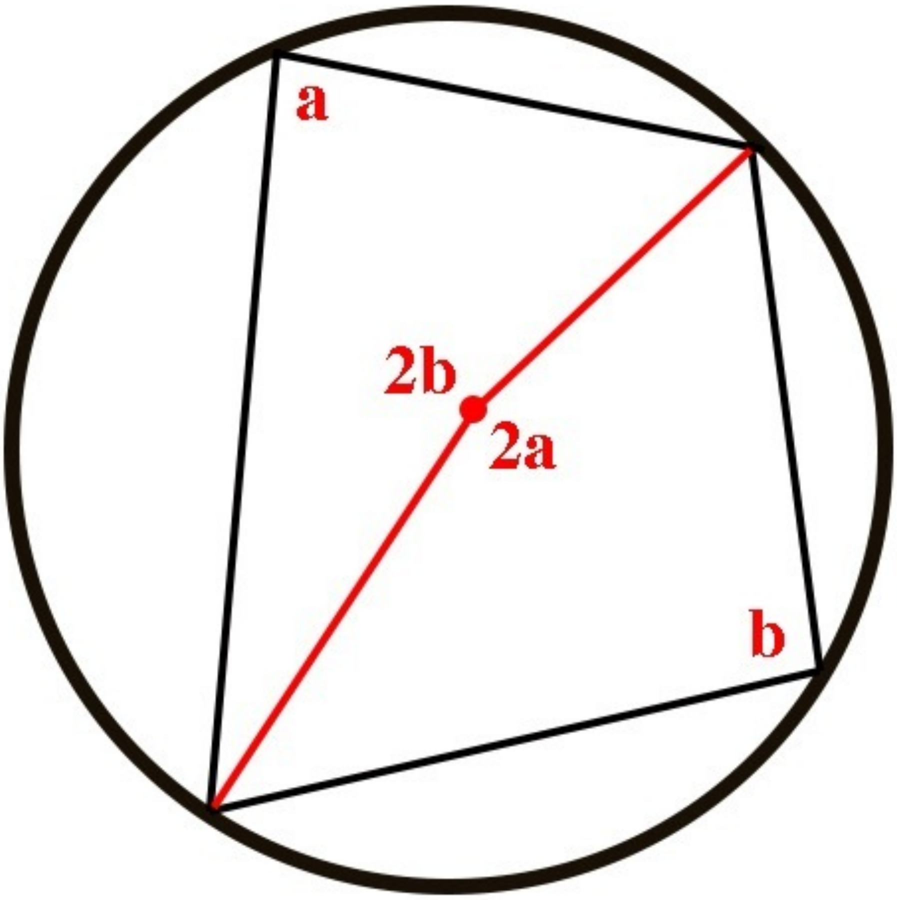
The sum of the opposite angles of a cyclic quadrilateral that has one diagonal as a diameter is …

90 + 90 = 180º

Opposite interior angles of a cyclic quadrilateral are labelled a and b.



To find the sum a + b we add lines to subtend angles at the centre.



Angles subtended at the circumference and the centre are labelled a and 2a, and b and 2b respectively.

Write down what we know …

$$2a + 2b = 360^o$$

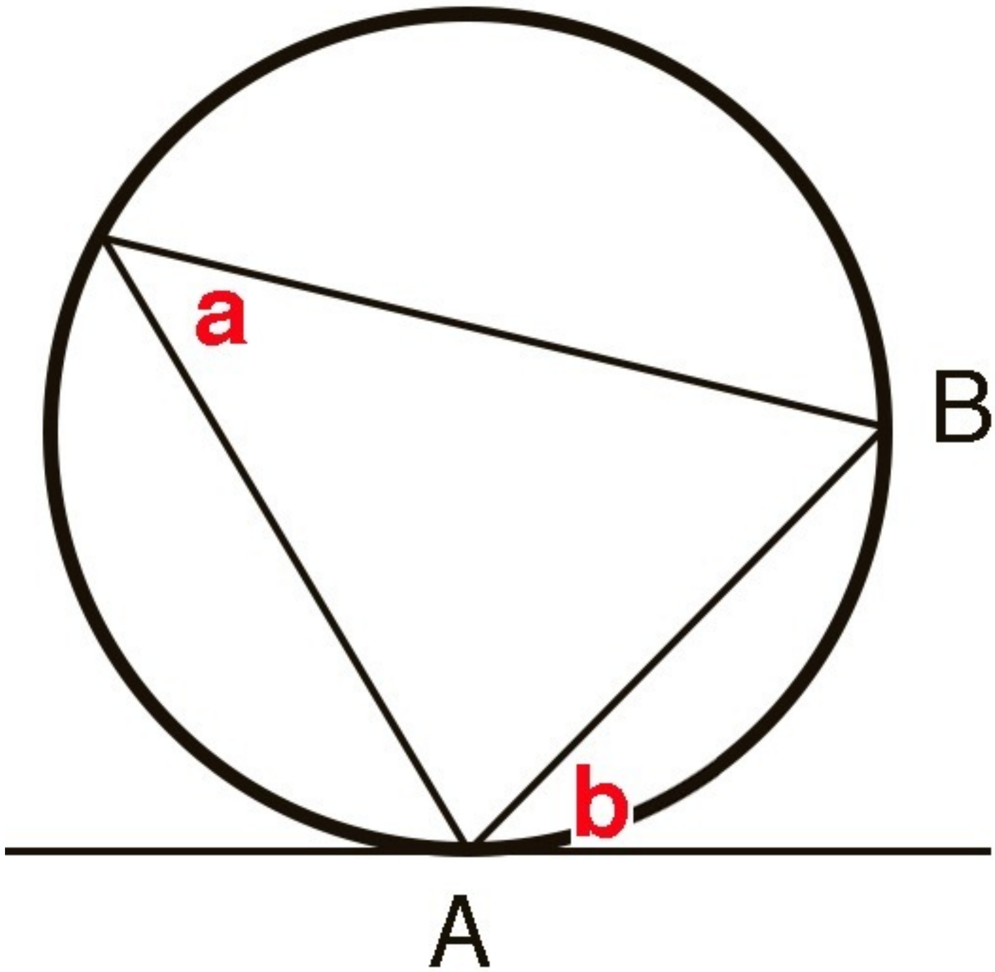
$$2(a + b) = 360$$

… and … $a + b = 180^o$

Since the diagram was not drawn in any special way the sum of the opposite angles of any cyclic quadrilateral must be 180°.

5 Angle between chord and tangent

A circle, tangent and chord AB are shown.



The tangent touches the circle at the point A.

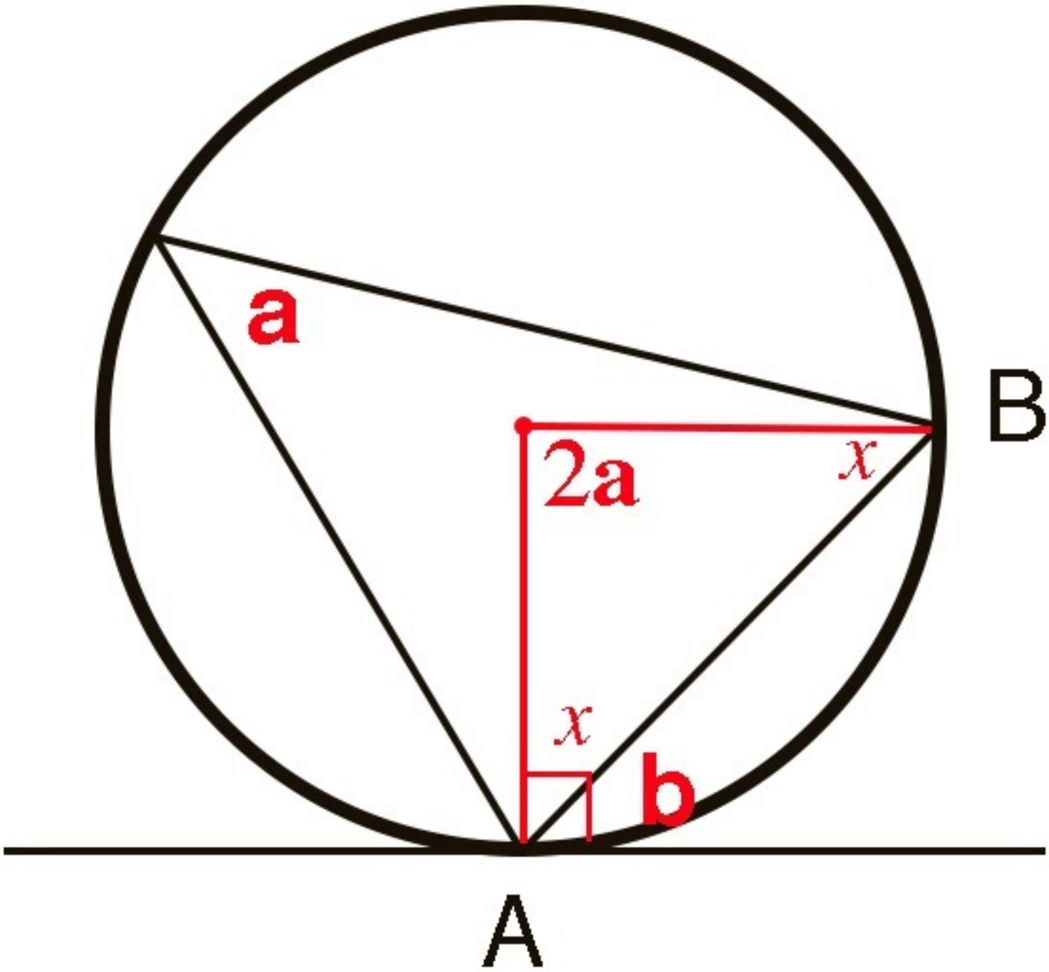
The angle subtended on AB (called a *chord* of the circle) is labelled a.

The angle between the chord AB and the tangent at A is labelled b.

Angle a and angle b look as though they might be equal.

To explore this idea we follow the procedure outlined four times above.

Lines are added to subtend an angle from AB to the centre of the circle.



Angles a and 2a are labelled, as are the equal angles (x) in the isosceles triangle. The radius through point A must intersect the tangent at right angles, which gives b = (90 - x).

Write down what we know …

$2a + 2x = 180^\circ$ … (angles of a triangle)

… so …. $a = (90 - x)$

… and … $a = b$ … (by inspection)

The diagram was not drawn in any special way. The result must be true for any angle between chord AB and tangent at A.