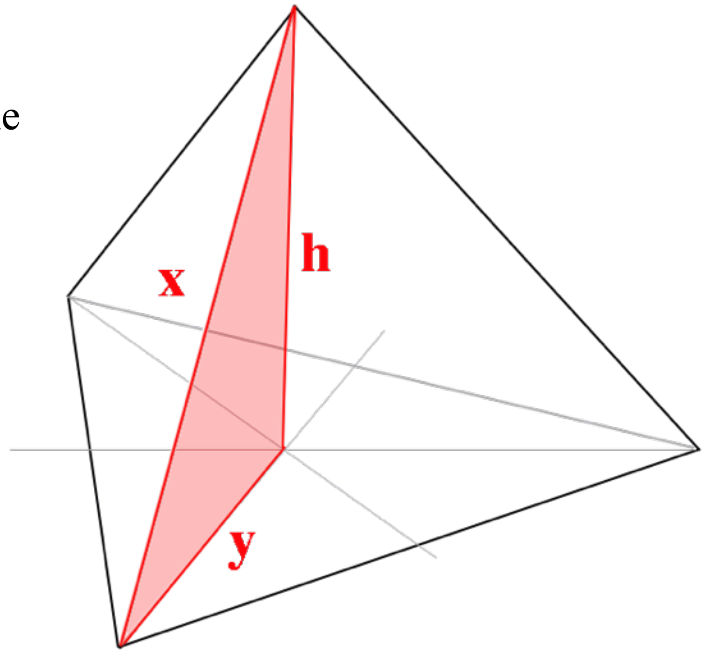
The Volume of a Regular Tetrahedron

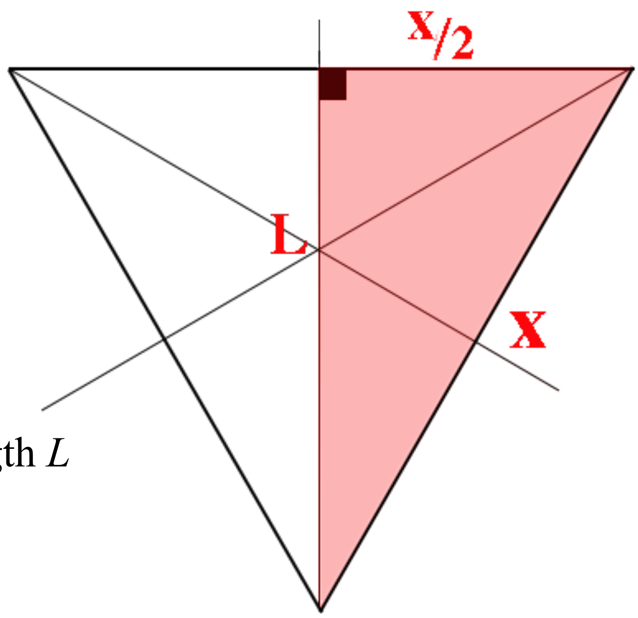
Shannon and Ian Jacobs

The area of an equilateral triangle x on a side is x^2 in triangular units. In square units the area of the base is ${}^{\sqrt{3}}/_{4}x^2$.

To find the height h we note that the top point of a regular tetrahedron is directly above the centroid of the base.



In the pink right angled triangle, which is the base of the tetrahedron…



$$x^2 = {}^{x^2}/_{4} + L^2$$

… and … $L = {}^{\sqrt{3}}/_{2}\,x$

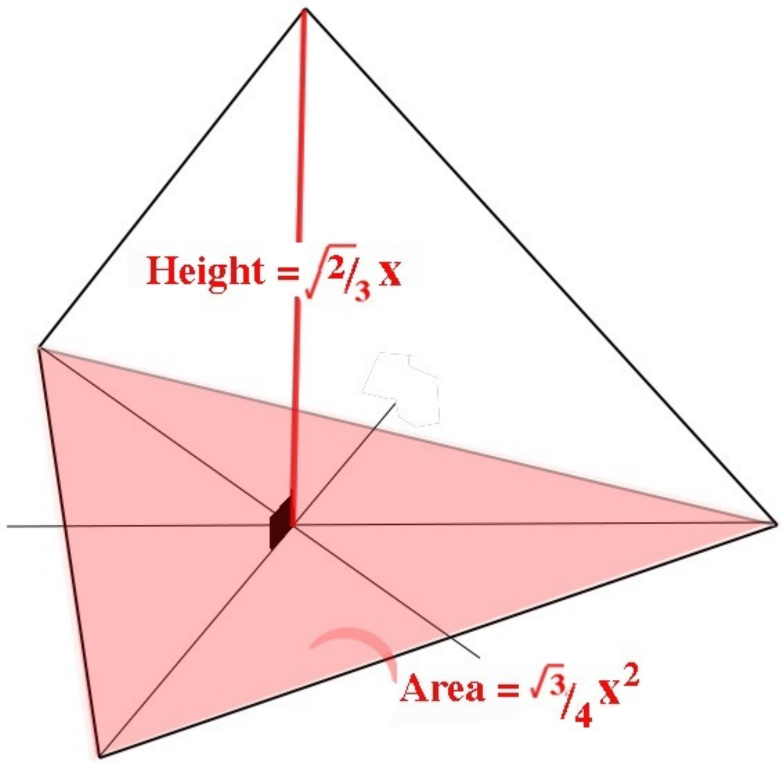
The centroid divides the median of length L in the ratio 2:1.

In the diagram above … $y = {}^{x}/_{\sqrt{3}}$

At once … $x^2 = {}^{x^2}/_{3} + h^2$

… and … $h = \sqrt{({}^{2}/_{3})}x$

The height and the area of the base are shown in the diagram below.



To calculate the volume of the regular tetrahedron we find one third of the area of the base times the height.

$$\text{Vol} = {}^1/_3 \times {}^{\sqrt{3}}/_4\, x^2 \times \sqrt{(^2/_3)}\, x$$

$$= {}^{\sqrt{2}}/_{12}\, x^3$$

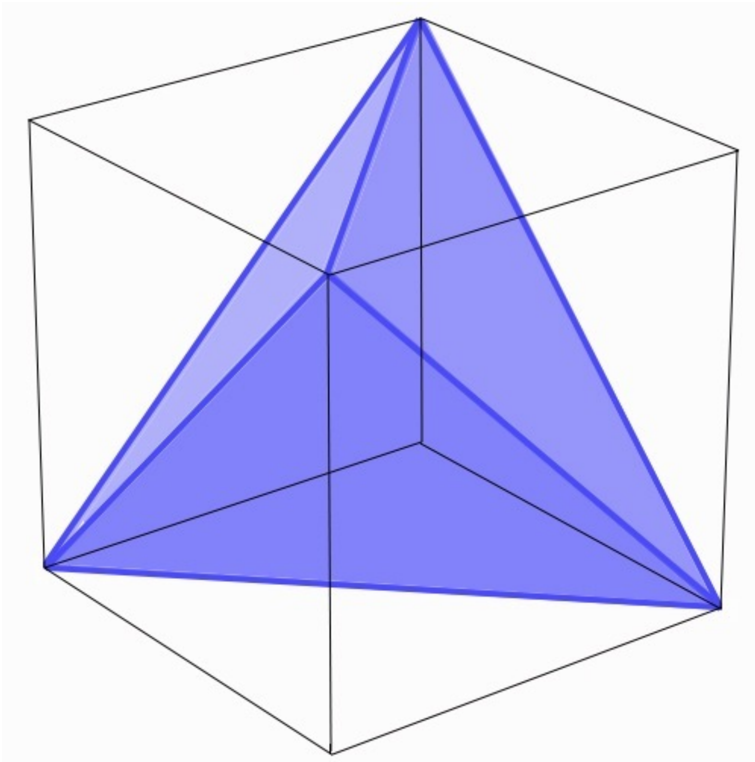
Multiplying top and bottom by √2 gives the usual form of this relationship.

$$\text{Vol} = \boldsymbol{{}^{x^3}/_{6\sqrt{2}}}$$

The volume has been found as efficiently as we can manage, using what we knew about the conversion of triangular to square units of area, the medians of a triangle, and the theorem of Pythagoras. Each of these is the subject of an article in this collection. The geometrical method below, which we find on the web, avoids the use of these results. *In mathematics there is usually another way to do anything*.

A recommended method

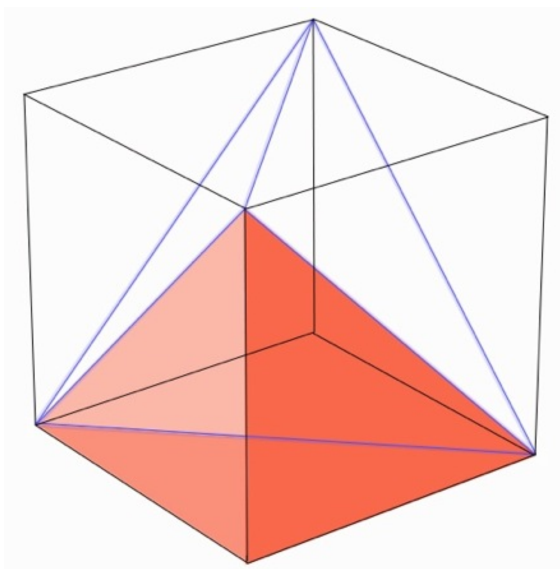
A cube has six faces. A tetrahedron has six edges. If we make the edges of a tetrahedron be the diagonals of the faces of a cube we have a tetrahedron surrounded by a cube.



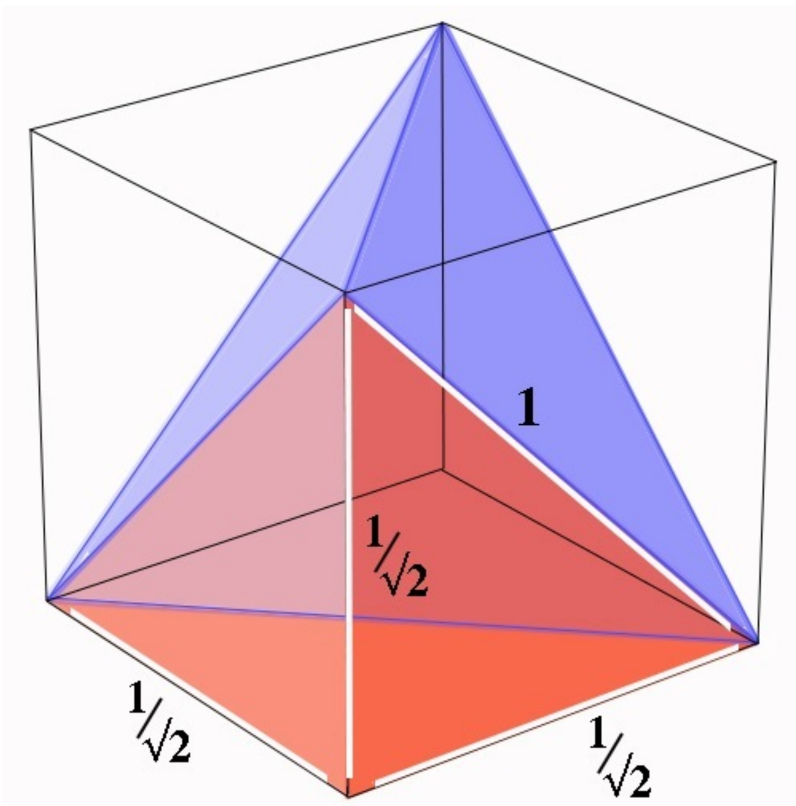
We know how to find the volume of a cube. If we can find the volume of one of the pyramids that have the corner of the cube as the apex we can find the volume of the tetrahedron.

Imagine the cube resting on a table. Look carefully at the pyramid that sits with the triangular base on the table. The base of this pyramid is a right angled isosceles triangle that has an area that we can calculate using half base times height.

The volume of this pyramid is also easy to calculate using one third the area of the base times the height, which is the edge of the cube.



Suppose the length of the edge of the tetrahedron is 1.



By the theorem of Pythagoras the edge of the cube has a length of $1/\sqrt{2}$.

The volume of the red pyramid is $1/3$ (the area of the base) x the height.

$$V = 1/3 \times (1/2 \times 1/\sqrt{2} \times 1/\sqrt{2}) \times 1/\sqrt{2}$$

$$= 1/12\sqrt{2}$$

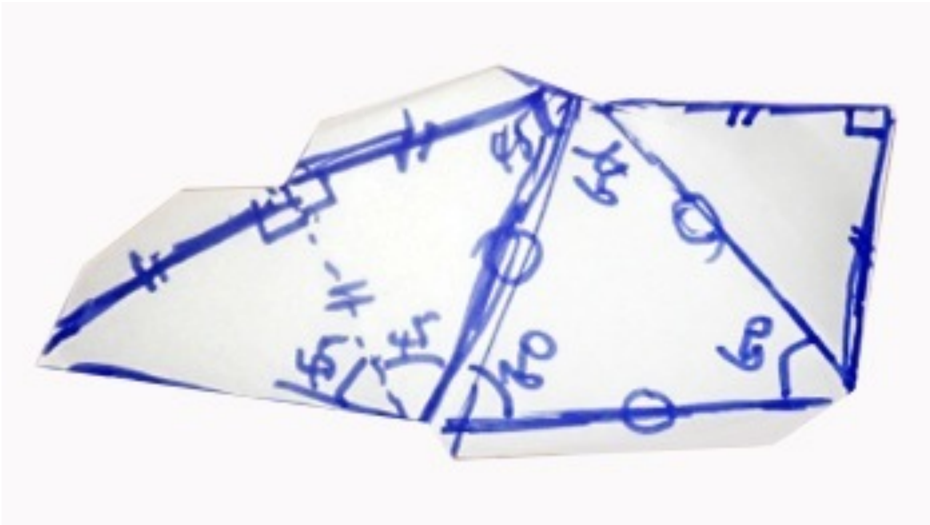
The volume of the tetrahedron is the volume of the cube, $1/2\sqrt{2}$, less 4V.

$$1/2\sqrt{2} - 4/12\sqrt{2} = 1/6\sqrt{2}$$

The derivation follows the same pattern if the edge length is *x*.

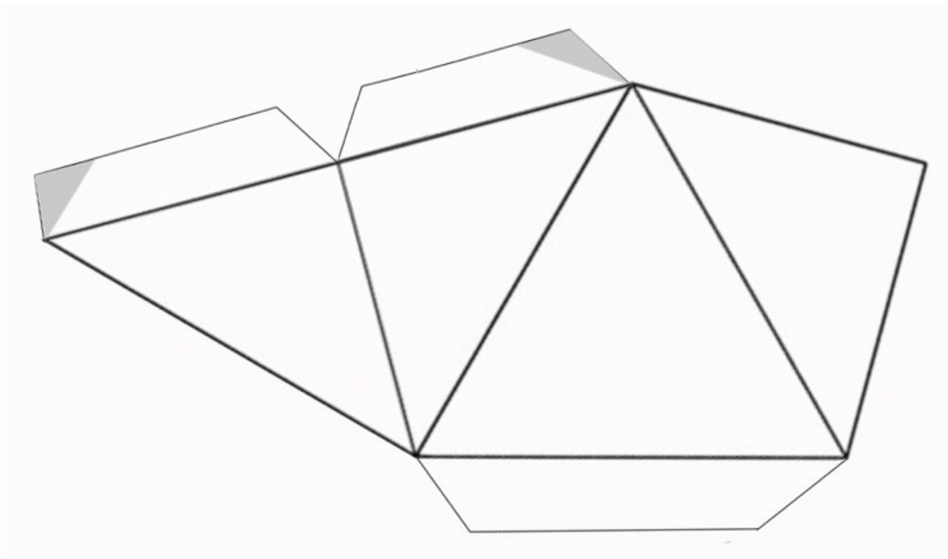
The volume of a tetrahedron of edge length *x* is $x^3/6\sqrt{2}$ as above.

We drew nets and made models of the five pyramids that go together to make the cube. There are two ways to make an unfamiliar net. Think about it carefully … or hack it with paper, scissors and a pen by just cutting and folding. Do it again a couple of times and draw angles and lengths on the result with a pen. Dad hacks them.



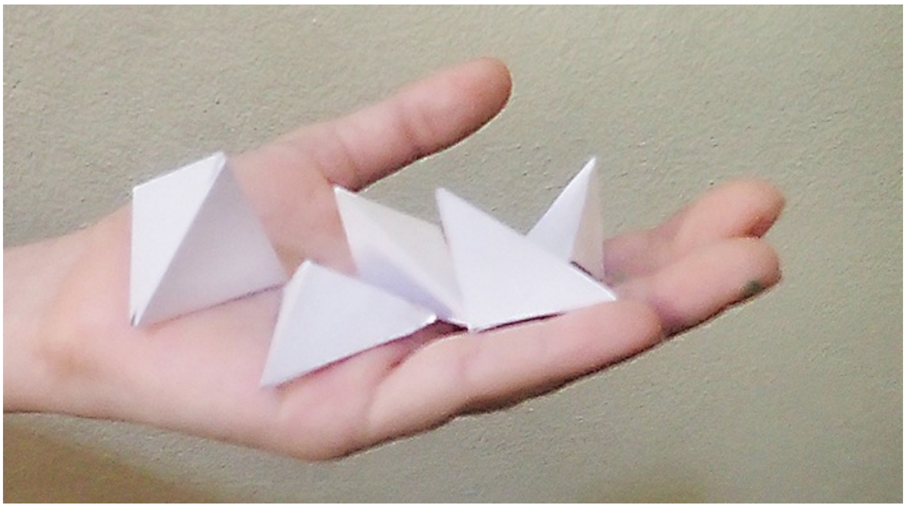
Thats it: an equilateral triangle and three right angled isosceles triangles.

Sheets of carefully drawn nets to make small models and larger versions are linked below.



Note: *the right angled pyramid nets are more difficult to fold than the net of a regular tetrahedron. The shaded triangle on two of the tags can be left in place to make the tags easier to fold down on the line. Once creased firmly, a tag can be folded up and the shaded corner can be removed with scissors.*

The completed models.



Putting them together makes a cube.