Area and Volume

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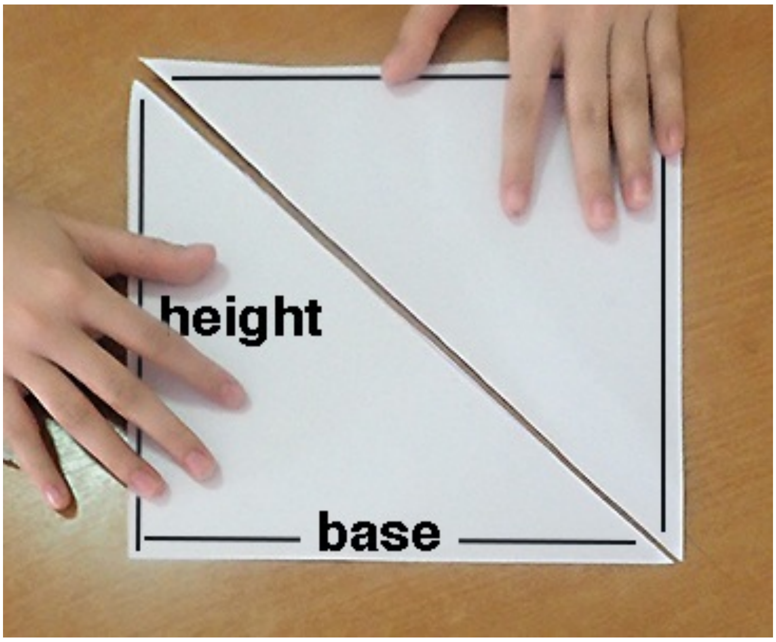
The area of a triangle

Fold the corner of a rectangular sheet of A4 copy paper and remove the end to leave a square piece of paper with a diagonal fold. Separate the square into two congruent (identical) triangles along the diagonal.

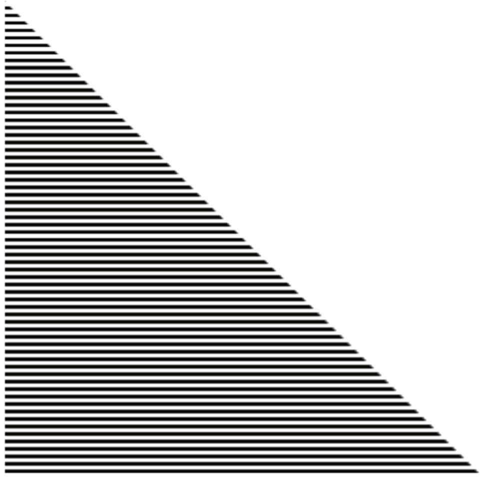


The area of a square is base times height, *bh*.

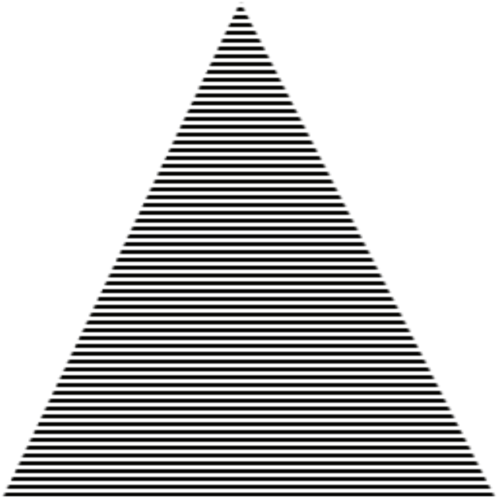
The area of the triangle is half base times height, $^1/_2\, bh$.

The area of *any* triangle is $^1/_2bh$

To prove this simply, without using a branch of mathematics called *Calculus*, imagine the right angled triangle above to be made from a stack of narrow lines. To make the diagram clear we will separate the lines, but imagine the lines to be touching, like a stack of playing cards or the pages of a book.



Label the lines from the base 1, 2, 3, …. n. Starting from the base, move each line a small distance n x p to the right. This process (*a shearing transformation*) changes the angles, the lengths of two sides, and keeps the sides straight.

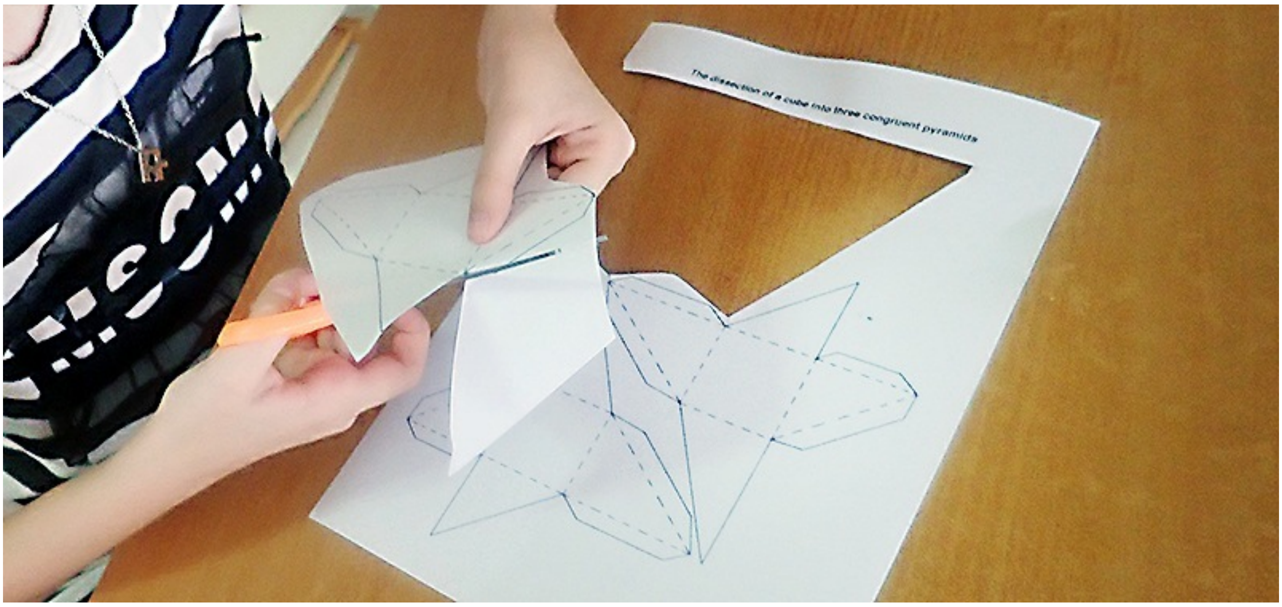


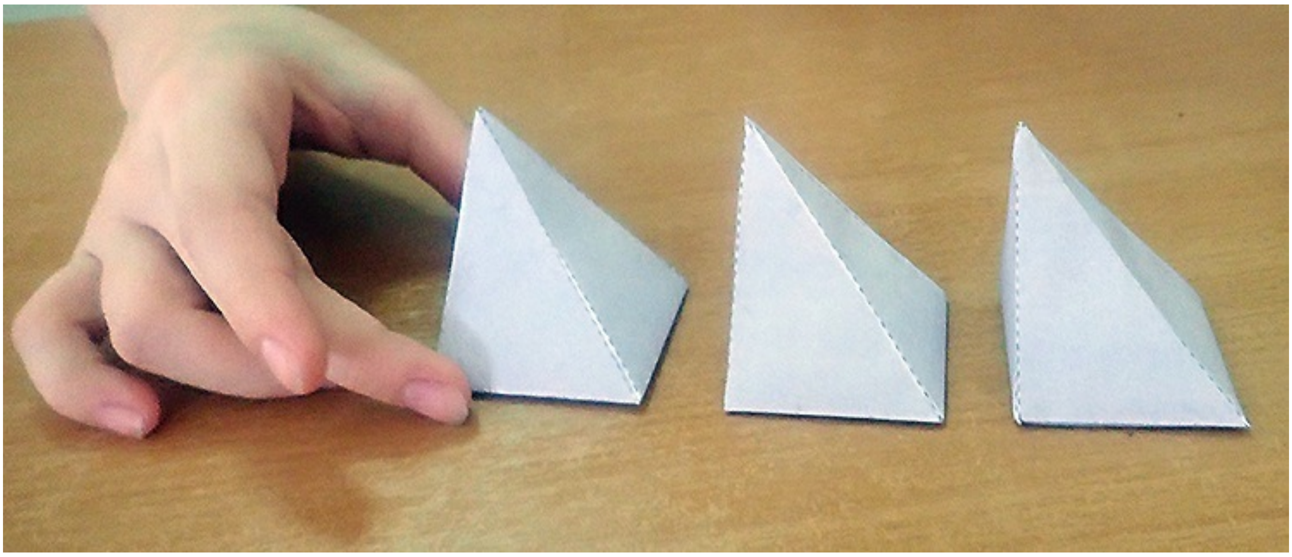
The area of the new triangle is made up of the same number of lines of the same lengths. The area must be the same. The base and height of the new triangle are unchanged, and the area is again given by $^1/_2bh$.

The volume of a pyramid

A cube can be made from three *congruent* (identical) pyramids. The pyramids each have a diagonal of the cube as their longest edge.

The nets in the image are linked for download below. They are cut, folded, and the tags glued inside the faces to make the three pyramids.





The pyramids are identical: they each have a square base of the same area and have the same height.

Putting the pyramids together carefully makes a cube.

The volume of a cube is (the area of the base) x (the height).

The volume of one of our three congruent pyramids is $^1/_3 Bh$, where B is the *area* of the base, and h is the vertical height of the top of the pyramid above the level of the base.

Imagining that the pyramid is made from planes like a deck of cards and shifting the cards with a shearing transformation, as we did for the lines of the triangle above, leaves the base and height of the pyramid unchanged. The volume of these pyramids made from the same sheets of card, before and after the shearing transformation, is unchanged. This shows at once that the volume of *any* pyramid is $^1/_3 Bh$.

Imagining that a pyramid was made from parallel planes that he called *indivisibles,* is known as Cavalieri's Principle. *Bonaventura Cavalieri* was an Italian who admired Galileo and wrote letters to him. He died in poor health in 1647. There is an article at the url below that you might like to read. It is full of the most wonderful sounding Italian names.

https://mathshistory.st-andrews.ac.uk/Biographies/Cavalieri/

Note: a circular cone is a pyramid with an infinite number of sides. The volume of a cone is $^1/_3 Bh$ where B is the area of the circular base. Areas of circles and ellipses, and the volume and surface area of a sphere, all of which involve the value of pi, 3.1419 … , are dealt with in a following article.