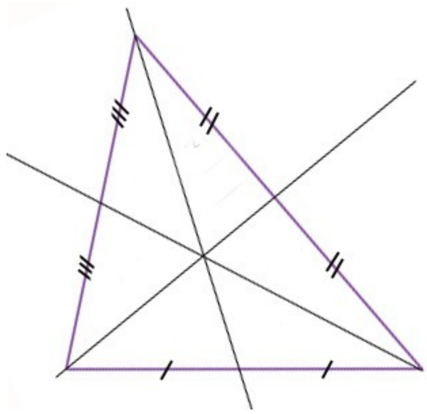
The medians of a triangle

Shannon and Ian Jacobs

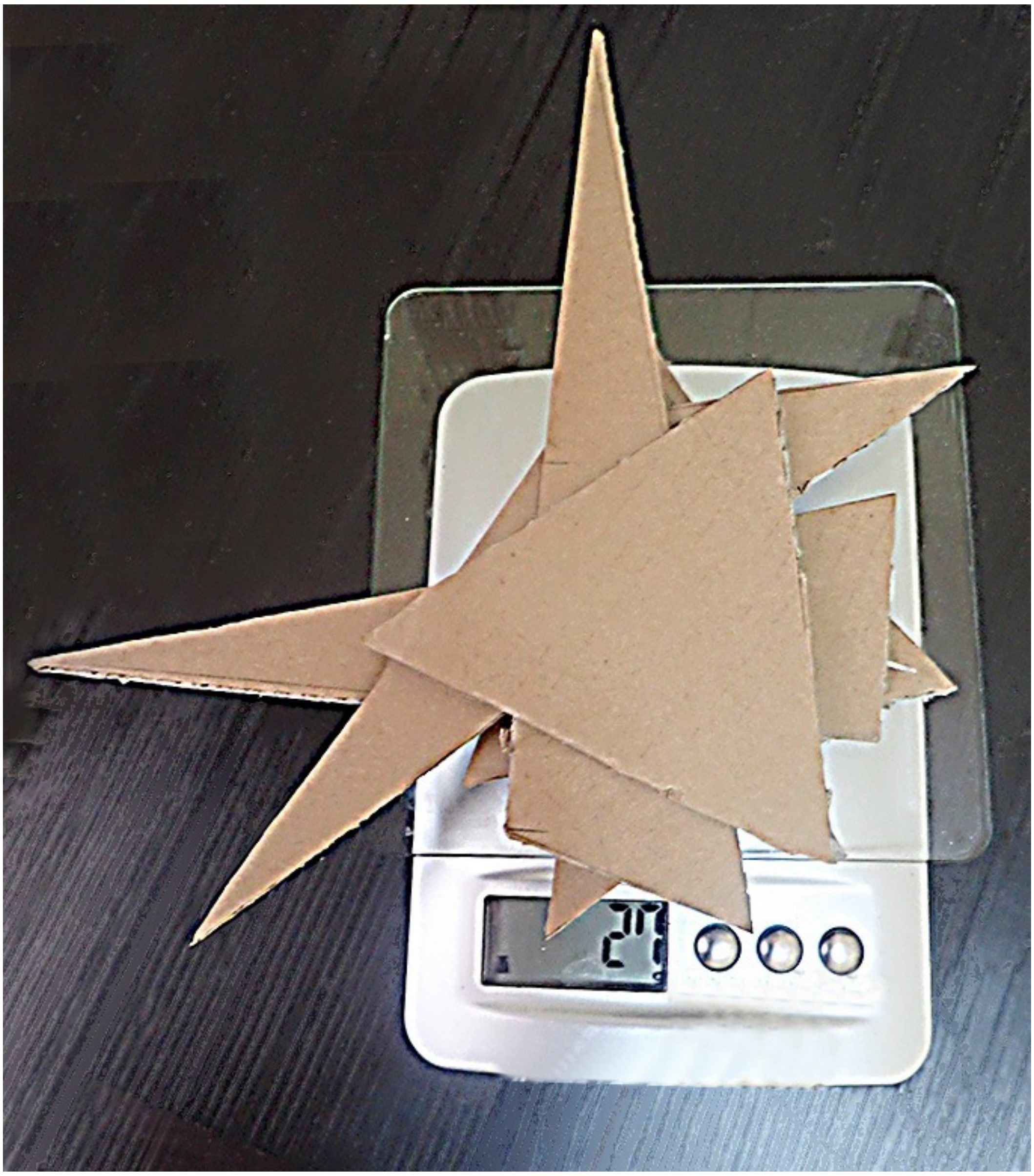


The three medians of a triangle meet at a point called the centroid. They cut the triangle into six small triangles that look as though they might have the same areas.

To find out if the areas are the same, without doing proofs, I drew a diagram on a full sheet of copy paper and cut it up.

I tried to weigh the paper triangles. The idea was good but I felt a bit silly because the balance we have reads grams and my pieces of paper were too light. Nothing happened, even with all the small triangles on at once.

To make heavier pieces I cut up a bigger triangle, that was drawn carefully on cardboard.



This time the six pieces together had a mass of 27 grams.

The scale readings for one triangle at a time were: 4, 4, 4, 4, 4 and 4 grams. The triangles were each about the same mass, but we needed a more sensitive balance to be sure.

To do that with this balance we hinged a long piece of expanded polystyrene from a door with tape to make a lever.

The more sensitive balance

The sensitivity of the balance was creased by a little more than five times. The scale readings were now: 24, 24, 23, 24, 23 and 24.

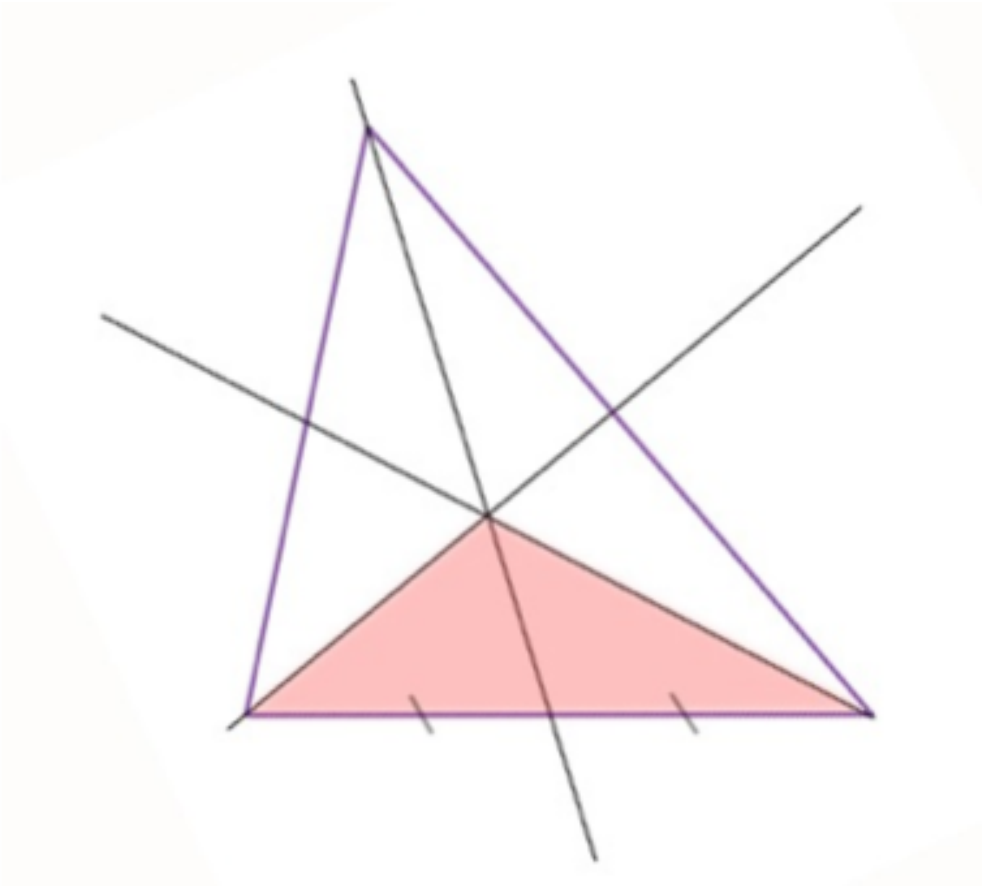
The small triangles had the same mass to within 5%. The areas were close to the same, but again we couldn't be sure that they were exactly equal.

> **Note:** *Finding areas by measurement in this way always involves uncertainties. If we bought a more sensitive balance, and had the LEGO people make triangles out of thick highly uniform plastic plate with machine tools, it might be possible to show that the areas of our six triangles were the same to within 0.001% but it's not possible to show in this way that they are exactly the same because in Physics, as in all of science, measurements can never be exact. Scientists learn to live with that. They estimate uncertainties and are careful to publish results that they claim to be reliable within stated limits.*

In the abstract world of mathematics diagrams can be drawn with exact relationships. Dad showed me how to prove mathematically that the six areas are exactly the same in a way that I can understand.

Equal areas, 2:1 ratio, and centre of mass

Cutting the six smaller triangles from a sheet of cardboard and weighing them showed that the areas were the same to within 5%.



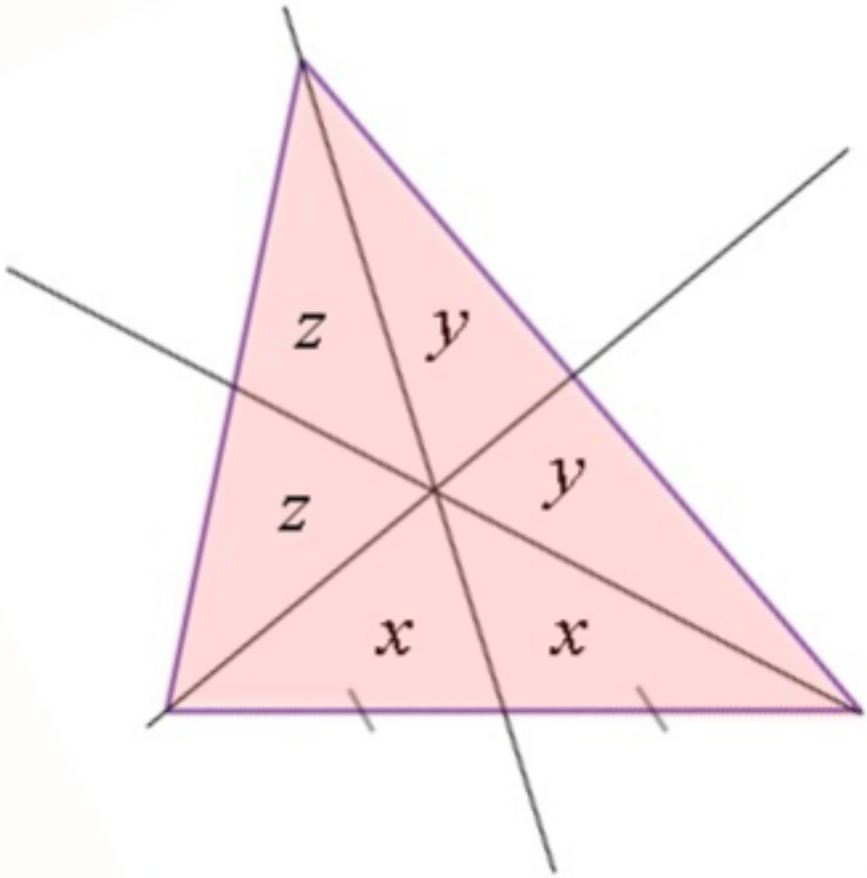
To show that the six small triangles in the diagram above, have exactly the same areas, and that that is true in any triangle in the abstract perfect world of pure mathematics, we use what we know about the area of a triangle.

Area = half (base) x (height)

The median drawn from the top point of the triangle to the lower side cuts that side at the mid point.

The base of each pink triangle must be the same length and the height of the two pink triangles is the same. The areas of the pink triangles are the same (*exactly the same*).

The area of each lower triangle is labelled x. Moving around the diagram and repeating the argument shows that each pair of triangles have the same areas, given the symbols, y and z on the diagram below.



To show that areas x, y and z are equal we do the same thing for the pair tall triangles on each side of the median drawn from the top point to the lower side as before. The base of each of these large triangles is the same: their height is the same: their areas are the same.

$$x + 2z = x + 2y$$

and … $\quad z = y$

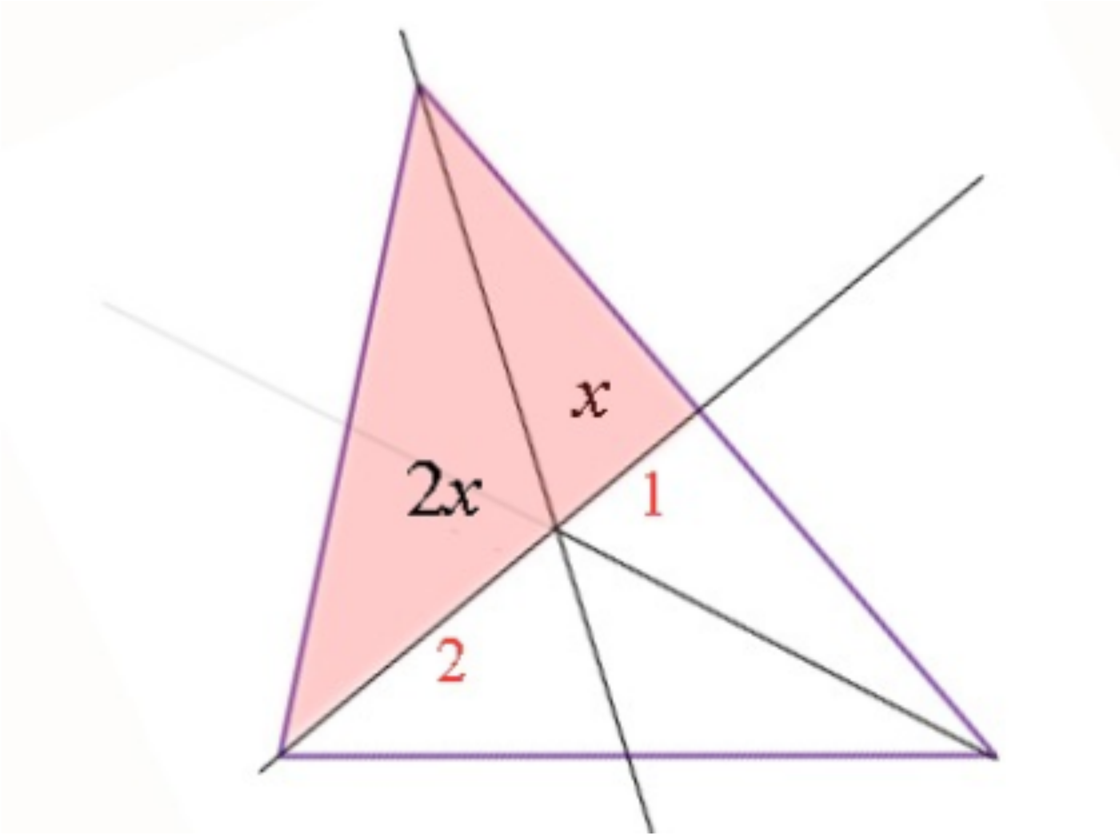
Moving around the diagram and doing the same thing shows that $x = z$

and … $\quad x = y = z$

The areas of the six triangles are the same (*exactly the same*).

Looking at the small triangles and measuring the lengths on the large diagram we made to cut up the triangles to weigh them suggests that the medians are divided in the ration 2:1.

To *prove* that the centroid divides the medians in the ratio 2:1 look carefully at the two pink triangles below.



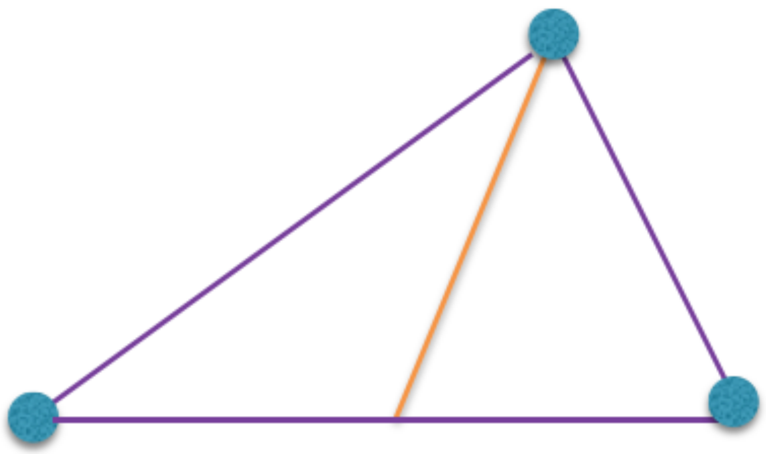
The two pink triangles that have areas $2x$ *and* x. They have the same height and their areas are given by half the base times the height.

The areas are in the ratio 2:1. The lengths of each base must be in the same ratio.

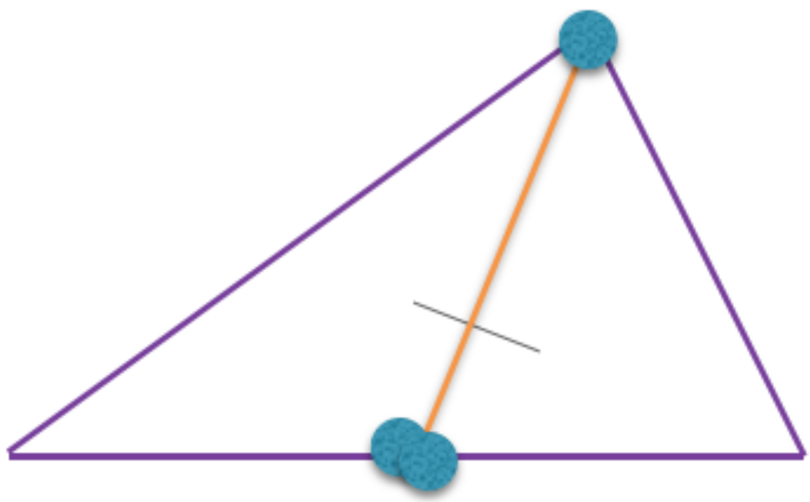
Moving around the diagram and repeating the argument shows that each median is divided in this ratio.

We found by drawing on cardboard that the medians crossed at about the same point (called the centroid) and that the centroid appeared to be the centre of mass, the point from which the triangle hung horizontally from a vertical string. Dad showed me how to prove these things for any triangle.

If a triangle of chipboard is hung horizontally from three vertical strings from the points, the weight on each string is the same. *You could do this for yourself*. We can replace a uniform triangle with three equal masses, one at each vertex.



The orange line from the vertex is a median. The centre of mass of the lower two masses is at the mid point of the base of the triangle. We can replace them with one mass of $2m$ at the mid-point of the base.



We now have one mass of m and one of $2m$ at each end of the median. The centre of mass is now the point on the median divided in the ratio 2:1. The same argument applies along each median. This proves that the medians meet at a single point (the centroid) and that the centroid is the centre of mass of a uniform triangle.